An Introduction to

Graph Theory and Complex Networks

PROBLEMS

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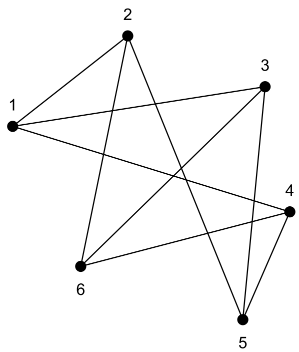
CONTENTS

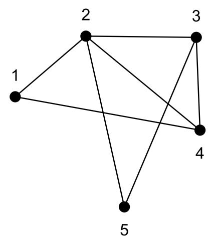
Problems Chapter 2

Q 1: Give the adjacency matrix for each of the following graphs, and draw those graphs.

G1: $V = \{1, 2, 3, 4, 5, 6\}$ and
$E = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 3,5\rangle, \langle 3,6\rangle, \langle 4,5\rangle, \langle 4,6\rangle\}$

G2: $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 3,5\rangle\}$





	1	*2*	*3*	*4*	*5*	*6*
1	*0*	*1*	*1*	*1*	*0*	*0*
2	*1*	*0*	*0*	*0*	*1*	*0*
3	*1*	*0*	*0*	*0*	*1*	*1*
4	*1*	*0*	*0*	*0*	*1*	*1*
5	*0*	*1*	*1*	*1*	*0*	*0*
6	*0*	*0*	*1*	*1*	*0*	*0*

	1	*2*	*3*	*4*	*5*
1	*0*	*1*	*0*	*1*	*0*
2	*1*	*0*	*1*	*1*	*1*
3	*0*	*1*	*0*	*1*	*1*
4	*1*	*1*	*1*	*0*	*0*
5	*0*	*1*	*1*	*0*	*0*

Q 2: Consider the following two graphs:

G1: $V = \{1, 2, 3, 4, 5, 6\}$ and
$E = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 3,5\rangle, \langle 3,6\rangle, \langle 4,5\rangle, \langle 4,6\rangle\}$

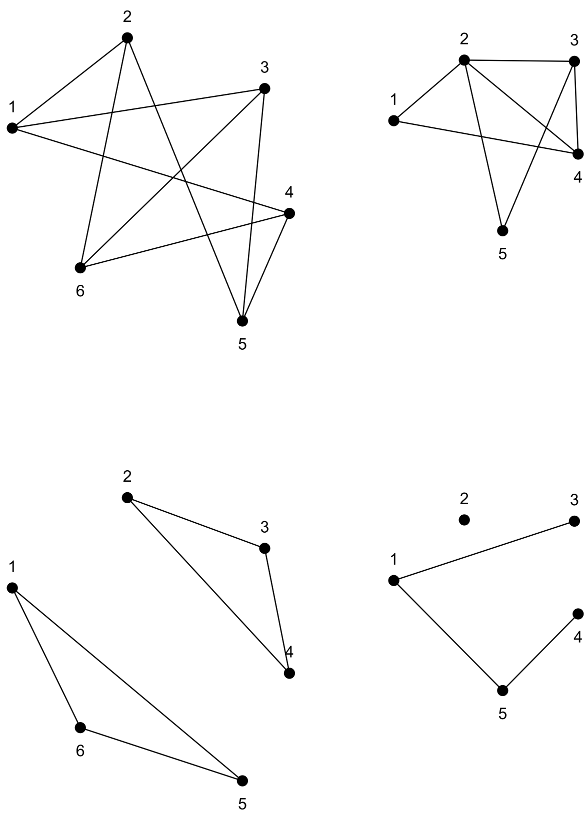
G2: $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 3,5\rangle\}$

For each graph, check whether it is (1) bipartite, (2) complete, (3) complete bipartite, (4) complete nonbaprtite.

It is not hard to see that G_1 is isomorphic to $K_{3,3}$, with $V_1 = \{2, 3, 4\}$ and $V_2 = \{1, 5, 6\}$. In contrast, G_2 neither complete, nor bipartite, nor complete bipartite.

To show that G_2 is not bipartite, we can, either exhaustively try all combinations of partitioning its vertex set, yet it is easier to simply identify an odd-length cycle (such as, for example, $[1, 2, 4, 1]$.

Q 3: Draw the complement of the following two graphs:



Q 4: Prove that for any graph, the sum of its vertex degrees is even.

A simple answer is that we already know that for a graph $G = (V, E)$, we have that $\sum_{v \in V} \delta(v) = 2 \cdot |E|$, which is obviously even. Alternatively, we can start counting the edges incident with each vertex, and the realizing that we count every edge twice.

Q 5: Show that every simple graph has two vertices of the same degree.

This can be shown using the pigeon hole principle. Assume that the graph has n vertices. Each of those vertices is connected to either $0, 1, 2, ..., n-1$ other vertices. If any of the vertices is connected to $n-1$ vertices, then it is connected to all the *others, so there cannot be a vertex connected to 0 others. Thus it is impossible to have a graph with n vertices where one is vertex has degree* 0 *and another has degree* $n-1$*. Thus the vertices can have at most* $n-1$ *different degrees, but since there are n vertices, at least two must have the same degree.*

Q 6: Show that if n people attend a party and some shake hands with others (but not with themselves), then at the end, there are at least two people who have shaken hands with the same number of people.

The solution is easy to see when modeling this situation as a graph with n vertices. Two vertices are joined if their associated people have shook hands. The question is then showing that at least two people have shook the same number of hands, or, in other words, that their vertex degree is the same.

Q 7: Show that if every component of a graph is bipartite, then the graph is bipartite.

Let G consist of components $H_i = (V_i, E_i)$*, with set* $V_i = V_{i,1} \cup V_{i,2}$*. Clearly, with* $V_1 = \bigcup_i V_{i,1}$ *and* $V_2 = \bigcup_i V_{i,2}$*, we have partitioned the vertex set of G such that no vertices from* V_1 *or from* V_2 *are joined.*

Q 8: Show that the complement of a bipartite graph need not to be a bipartite graph.

Simply consider $K_{3,3}$*, whose complement is certainly not bipartite because of the existence of an odd-length cycle.*

Q 9: Prove the following. Consider a list $\mathbf{s} = [d_1, d_2, \ldots, d_n]$ of n numbers in descending order. This list is graphic if and only if $\mathbf{s}^* = [d_1^*, d_2^*, \ldots, d_{n-1}^*]$ of $n-1$ numbers is graphic as well, where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for } i = 1, 2, \ldots, d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

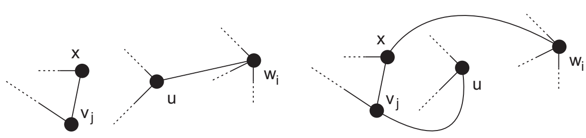
Let us first assume that $\mathbf{s}^*$ *is graphic. We then need to show that* $\mathbf{s}$ *is also graphic. Let* G^* *be a simple graph with degree sequence* $\mathbf{s}^*$*. We now construct a simple graph G from* G^* *with degree sequence* $\mathbf{s}$ *as follows (and in doing so, we show that* $\mathbf{s}$ *is graphic). Take* G^* *and add a vertex u. For readability, let* $k = d_1$ *and consider the k vertices* $v_1, v_2, \ldots, v_k$ *from* G^* *having respectively degree* $d_1^*, d_2^*, \ldots, d_k^*$*. We then join these vertices to the newly added vertex u. Obviously, u now has degree k, but also each vertex* v_i *now has degree* $d_i^* + 1$*. Because all other vertices of* G^* *are not joined with u, their vertex degree is left unaffected. As a consequence, the newly constructed graph G has degree sequence* $[k, d_1^* + 1, d_2^* + 1, \ldots, d_k^* + 1, d_{k+1}^*, \ldots, d_{n-1}^*]$*, which is precisely* $\mathbf{s}$*.*

Let us now consider the opposite: if $\mathbf{s}$ *is graphic, we need to show that* $\mathbf{s}^*$ *is so as well. In other words, we need to find a graph* G^* *that has degree sequence* $\mathbf{s}^*$*. To this end, we consider three different sets of vertices from* G*. Let* u *be a vertex with degree* $k = d_1$*. Let* $V = \{v_1, v_2, \ldots, v_k\}$ *be the respective vertices with the* k *next highest degrees* $d_2, d_3, \ldots, d_{k+1}$*. Finally, let* $W = \{w_1, w_2, \ldots, w_{n-k-1}\}$ *be the remaining* $n-k-1$ *vertices with degree* $d_{k+2}, d_{k+3}, \ldots, d_n$*, respectively.*

Consider the graph G^* *by removing* u *from* G*, along with the* k *edges incident with* u*. If each of these edges is incident with one of the vertices from* V*, then obviously* G^* *is a graph with degree sequence* $(d_2 - 1, d_3 - 1, \ldots, d_{k+1} - 1, d_{k+2}, \ldots, d_n)$*, which is precisely* $\mathbf{s}^*$*.*

Now consider the situation that u *is adjacent to a vertex from* W*, say* w_i*. If for some vertex* $v_j \in V$*, the degree of* v_j *and* w_i *are the same, i.e.,* $\delta(w_i) = \delta(v_j)$*, then we can simply swap* w_i *and* v_j *in the original construction of the sets* V *and* W*, meaning that* $\langle u, w_i \rangle$ *is now an edge incident with a vertex from* V *instead of* W*. However, if* $\delta(w_i) < \delta(v_k)$ *(i.e.,* $\delta(w_i)$ *is less than the degree of any vertex from* V*) we cannot apply such an exchange.*

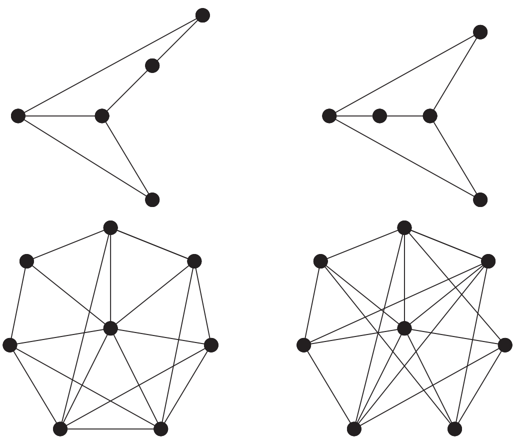
The problem that we need to solve is that there is now a vertex v_j *not adjacent to* u *whose degree will remain the same instead of being decremented by 1. Likewise, by simply removing* u *we would decrease the degree of* w_i*, while we would like to see it unaffected if we want to realize the degree sequence* $\mathbf{s}^*$*. Note, however, that because* $\delta(v_j) > \delta(w_i)$*, there is a vertex* x *adjacent to* v_j *but not adjacent to* w_i *(note also that* $x \neq u$*), as shown in (a) below. In constructing* G^* *we now first remove edges* $\langle u, w_i \rangle$ *and* $\langle v_j, x \rangle$*, and then add edges* $\langle x, w_i \rangle$ *and* $\langle u, v_j \rangle$*, leading to the situation shown in (b) below. The effect is that we now have a graph* G' *in which* u *is adjacent to* v_j *instead of* w_i*, but without affecting the degree of* u*,* v_j*,* x*, or* w_i*. In other words,* G' *has the degree sequence* $\mathbf{s}$*. If* u *is now adjacent to vertices only from* V*, we have already shown that* $\mathbf{s}^*$ *is graphic. If* u *is still adjacent to a vertex from* W*, we apply the same method to construct a graph* G'' *in which* u *is adjacent to one more vertex from* V*. If necessary, we repeat this method until* u *is adjacent only to vertices from* V*, at which point we know that* $\mathbf{s}^*$ *is graphic.*



(a) *(b)*

Q 10: Show that two graphs with the same degree sequence need not be isomorphic.

Counter examples include the following:



Q 11: Show that there is no simple graph with 12 vertices and 28 edges in which

(a) the degree of each vertex is either 3 or 4, or

(b) the degree of each vertex is either 3 or 6.

Suppose there is a graph with k vertices of degree 3 in the graph. For (a), if the remaining $(12-k)$ *vertices have all degree 4, the equation* $3k + 4(12-k) = 56$ *gives* $k = -8$*, which is impossible. For (b), if the remaining* $(12-k)$ *vertices all have degree 6, the equation* $3k + 6(12-k) = 56$ *gives* $k = 5\frac{1}{3}$*, which is also not possible.*

Q 12: Show that there is no simple graph with four vertices such that three vertices have degree 3 and one vertex has degree 1.

Suppose that such a graph G exist. We know that $|E(G)| = 5$*. Let* $v \in V(G)$ *with* $\delta(v) = 1$*. Consider the graph* $G_1 = G - v$*, having 3 vertices and 4 edges, one vertex w having degree 2. Let* $G_2 = G_1 - w$*. Clearly,* G_2 *has 2 vertices and 2 edges. In order for G to be simple,* G_2 *must be simple as well. This is impossible.*

Q 13: Show that the number of vertices in a k-regular graph is even if k is odd.

Recall that in a k-regular graph, each vertex has degree k. We know that $\sum \delta(v) = k \cdot n = 2 \cdot m$*, where m is the number of edges and n the number of vertices. In other words,* $k \cdot n$ *must be even, which is possible only if n is even when k is odd.*

Q 14: Let $v = [d_1, d_2, \ldots, d_n]$ and $w = [w_n, w_{n-1}, \ldots, w_2, w_1]$, where $w_i = n - 1 - d_i$. Show that v is graphic if and only if w is graphic.

Suppose v is the degree sequence of $G = (V, E)$*, where* $V = \{l, 2, ..., n\}$*. It is easy* *to see that w is the degree vector of the complement of G. Thus v is graphic if and only if w is graphic.*

Q 15: Show that there is no simple graph with six vertices of which the degrees of five vertices are 5, 5, 3, 2, and 1.

Suppose there is a simple graph G, and let k be the degree of the sixth vertex. The sum of the 6 degrees has to be even and $k \leq 5$. As a consequence, $k \in \{0, 2, 4\}$. If $k = 0$, then the degree sequence of G is $\mathbf{d} = [5, 5, 3, 2, 1, 0]$. However, $\mathbf{d}$ is not graphic. If $k = 2$, we have $\mathbf{d} = [5, 5, 3, 2, 2, 1]$, which is also not graphic. Finally, with $k = 4$ we have $\mathbf{d} = [5, 5, 4, 3, 2, 1]$, which is, again, not graphic.

Q 16: Find k if $[8, k, 7, 6, 6, 5, 4, 3, 3, 1, 1, 1]$ is graphic.

We need merely test the cases $k = 8$ and $k = 7$. It turns out that only for $k = 7$ we have a graphic sequence.

Q 17: Show that an ordered sequence of nonincreasing numbers in which no two numbers are equal cannot be graphic.

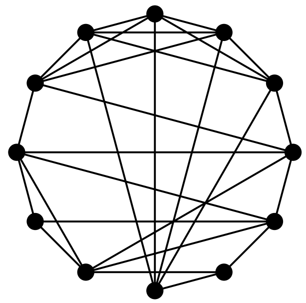
Consider the sequence $\mathbf{d}$ with k elements in which each element is a nonnegative integer. If no two elements are equal, $\mathbf{d} = [k-1, k-2, \ldots, 1, 0]$. Simply removing the first element and subtracting 1 from the rest, leaves us with -1 for the last element. Therefore, $\mathbf{d}$ cannot be graphic.

Q 18: Show that in a simple graph, there are at least two vertices with equal degrees.

By contradiction: if no two vertices have the same degree, the degree sequence will consist of numbers that are strictly decreasing. Such a sequence is never graphic.

Q 19: Show that there exists a simple graph with 12 vertices and 28 edges such that the degree of each vertex is either 3 or 5. Draw this graph.

We first compute how many vertices would have degree 3. Let this be k. We then know that $\sum \delta(v) = k \cdot 3 + (12 - k) \cdot 5 = 2 \cdot 28 = 56$. This gives us $k = 2$. The degree sequence of G is thus equal to $[5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 3, 3]$. Using the Havel-Hakim algorithm, we indeed find that this sequence is graphic. A graph corresponding to this sequence is the following.



Q 20: Show that there exists a simple graph with seven vertices and 12 edges such that the degree of each vertex is 2 or 3 or 4.

Suppose there are k vertices of degree 2 and l vertices of degree 3. Then the only solution in positive integers of the equation $2k + 3l + 4(7 - k - l) = 24$ *is* $k = 1$ *and* $l = 2$*. Thus if there is a graph with the desired property, it should have one vertex of degree 2, two vertices of degree 3, and four vertices of degree 4, giving a unique degree sequence* $\mathbf{d} = [4, 4, 4, 4, 3, 3, 2]$*. It is easily verified that this sequence is indeed graphic.*

Q 21: Prove that if u is a vertex of odd degree in connected graph G, then there exists a path from u to another vertex v of G where v also has odd degree.

There are several ways to prove this theorem. Because we know that the number of odd-degree vertices is even, and because G is connected, there is at least one other vertex v with odd degree, and thus an (u, v)*-path between the two.*
Another proof is by constructing the longest (u, v)*-path originating in u to any other vertex v in G. Then consider the graph* $g^* = G - P$*. Because the degree of u was odd, it will be even in* G^**. Likewise, because we are removing two edges for each intermediate vertex w on P, if the degree of w was odd in G, it will remain odd in* G^**, and likewise, remains even if it was even in G. If the degree of v was even, it will become odd, implying that the number of odd-degree vertices in* G^* *will increment by one, in turn meaning that* G^* *would have an odd-numbered of odd-degree vertices. That is not possible, and we conclude that the degree of v must have been odd.*

Q 22: Let $d(u, v)$ denote the length of the shortest (u, v)-path in a connected graph G. Prove that d satisfies the triangle inequality: for any $u, v, w \in V(G) : d(u, v) + d(v, w) \geq d(u, w)$.

Let P be a shortest (u, v)*-path and Q a shortest* (v, w)*-path. Clearly, a shortest*

(u,w)-path will have the same or less edges than the concatenation of P and Q, meaning that $d(u,w) \leq d(u,v) + d(v,w)$.

Q 23: Show that every simple graph with n vertices is isomorphic to a subgraph of the complete graph K_n.

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let K_n have vertices $\{u_1, u_2, \ldots, u_n\}$. Construct the subgraph $H \subseteq K_n$ such that $\langle u_i, u_j \rangle \in E(H)$ iff $\langle v_i, v_j \rangle \in E(G)$. Clearly, H is isomporphic to G.

Q 24: Prove that if two graphs G and G^* are isomorphic, then their respective ordered degree sequences should be the same.

Let ϕ be the one-to-one mapping by which G and G^* are known to be isomorphic. Consider vertex u from G and its adjacent vertices $v_1, \ldots, v_k$. By definition, each edge $e_i = \langle u, v_i \rangle$ incident with u in G is mapped to a unique edge $e_i^* = \langle \phi(u), \phi(v_i) \rangle$ in G^*. Because each edge e_i^* is incident with $\phi(u)$, we must have that $\delta(u) \leq \delta(\phi(u))$.

Now consider a vertex $v^* \in V(G^*)$ that is adjacent to $\phi(u)$. By definition of isomorphism, we know that the edge $e^* = \langle \phi(u), v^* \rangle$ must uniquely map to an edge $e = \langle \phi^{-1}(\phi(u)), \phi^{-1}(v^*) \rangle$ in G, where ϕ^{-1} denotes the inverse mapping of ϕ. Because ϕ is a one-to-one mapping, we also know that $\phi^{-1}(\phi(u)) = u$, and thus that $e = \langle u, \phi^{-1}(v^*) \rangle$. In other words, every edge incident with $\phi(u)$ in G^* will be incident with u in G. This means that $\delta(\phi(u)) \leq \delta(u)$.

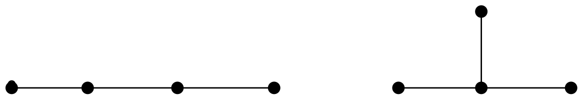
We conclude that $\delta(u) = \delta(\phi(u))$ for all vertices of G, implying that the ordered degree sequences of G and G^* should be the same.

Q 25: Show that if two graphs $G = (V, E)$ and $G^* = (V^*, E^*)$ are isomorphic, then $|V| = |V^*|$ and $|E| = |E^*|$.

If G and G^* are isomorphic, there is a one-to-one mapping $\phi : V \to V^*$ such that for every edge $e \in E$ with $e = \langle u, v \rangle$, there is a unique edge $e^* \in E^*$ with $e^* = \langle \phi(u), \phi(v) \rangle$. This means that $|E| \leq |E^*|$, but because ϕ is one-to-one, we also have that ϕ^{-1} will map every edge of E^* to a unique edge in E. Therefore, $|E| = |E^*|$. Because each edge is uniquely mapped, we also have that $|V| = |V^*|$.

Q 26: Show that two graphs G and $G*$ each having n vertices and m edges, need not be isomorphic.

The following two graphs can never be isomorphic, yet have the same number of vertices and the same number of edges.



Q 27: Show that two simple graphs are isomorphic if and only if their complements are isomorphic.

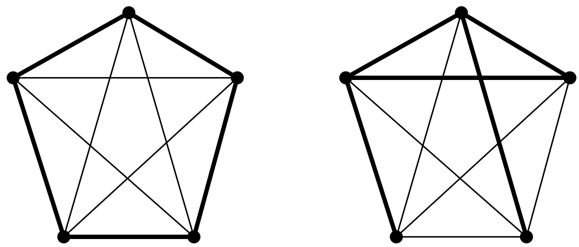
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs isomorphic under the one-to-one mapping ϕ. Extend G_1 by adding an edge $e = \langle u, v \rangle$ between two nonadjacent vertices, along with the edge $e' = \langle \phi(u), \phi(v) \rangle$. Obviously, $\phi(u)$ and $\phi(v)$ were also nonadjacent in G_2, and again both graphs $G_1 + e$ and $G_2 + e^*$ are isomorphic under ϕ. Continue until G_1 and G_2 have been extended to a complete graph G_1^* and G_2^*, respectively. Clearly, $\overline{G_1} \stackrel{\text{def}}{=} K_n - G_1$ is isomorphic with $G_1^* - G_1$, which in turn is isomorphic with $G_2^* - G_2$.

Q 28: Find a self-complementary graph G having four vertices.

The number of edges in the complete graph with four vertices is 6. So if G is a self-complementary graph with four vertices, it should have three edges. Consider the following two versions of K_4, each with a thick-edged subgraph, and its thin-edged complement. Clearly, only the second one is self-complementary.

Q 29: Find two self-complementary graphs having five vertices.

We know that $|E(K_5)| = \binom{5}{2} = 10$. In other words, the graph we are looking for should have five edges. Consider the following two versions of K_5, each with a thick-edged subgraph H and its thin-edged complement. In both cases, H is self-complementary.



Q 30: Prove by induction that a complete graph with n vertices contains $n(n-1)/2$ edges.

Obviously, for K_1 the statement is true. Assume the that it holds for the complete graph on $k > 1$ vertices. Consider K_{k+1} and remove an arbitrary vertex v. Clearly, $K_{k+1} - v$ is isomorphic with K_k, and therefore has $k(k-1)/2$ edges. Vertex v had degree k, so that the total number of edges in K_{k+1} is equal to $k(k-1)/2 + k = (k+1)k/2$.

Q 31: Compute the number of edges in K_n and in $K_{m,n}$.

There are different ways to do this. First, there are exactly $\binom{n}{2}$ ways of joining two vertices in K_n, which is exactly $n(n-1)/2$. Alternatively, let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. We have $n-1$ vertices to join v_1 to, a remaining $n-2$ for v_2, and, in general, a remaining $n-i$ for v_i. Therefore, the total number of edges is $(n-1) + (n-2) + \cdots + 1 = n(n-1)/2$.
For $K_{m,n}$ with partitioned vertex set $V(K_{m,n}) = V_1 \cup V_2$, we need merely consider the, say, m elements of V_1. Each vertex can be joined with one of the n vertices of V_2, leading to a total of $|V_1| \cdot n = m \cdot n$ edges.

Q 32: Use the fact that $\sum \delta(v) = 2|E|$ to find the size of K_n and $K_{m,n}$.

Let the number of edges of K_n be M. The degree of each vertex is $(n-1)$. There are n vertices. Thus the sum of the degrees of the n vetices is $n(n-1)$, which is $2M$. Hence $M = n(n-1)/2$. For $K_{m,n}$ let the number of its edges be M. Assume $V(K_{m,n}) = V_1 \cup V_2$, with $|V_1 = m|$ and $|V_2 = n|$. The degree of each vertex in V_1 is n, and the degree of each vertex in V_2 is m. The sum of the degrees of the m vertices in V_1 is therefore $m \cdot n$, while the sum of the degrees of the n vertices in V_2 is also $n \cdot m$. Therefore, $\sum \delta(v) = 2M = 2 \cdot m \cdot n$, so that $M = m \cdot n$.

Q 33: Show that $(n-1) + (n-2) + (n-3) + \cdots + 1 + 0 = n(n-1)/2$

The easiest way to do this is by induction, starting with $n = 1$, for which equality clearly holds. Assume it also holds for any $k > 1$. Consider the case $k+1$. We then *have:*

$$\begin{aligned}&(k+1)-1+(k+1)-2+\cdots+1+0=\\&k+\big((k-1)+(k-2)+\cdots+1+0\big)=\\&k+k/(k-1)/2=\\&(k+1)k/2\end{aligned}$$

Q 34: Show that the number of vertices in a self-complementary graph is either $4k$ or $4k+1$, where k is a positive integer.

Consider a self-complementary graph $G=(V,E)$ with n vertices and m edges. Since G is isomorphic to its complement, both G and its complement have the same number of edges. Now every edge in the complete graph with V as the set of vertices is either an edge in G or an edge in its complement. Thus $m+m=n(n-1)/2$, showing that $n(n-1)=4k$, where k is a positive integer. So $n=4k$ or $4k+1$.

Q 35: Show that every graph has an even number of odd-degree vertices.

Suppose the sum of the degrees of the odd vertices is k and the sum of the degrees of the even vertices is l. The number l is even, and the number $k+l$, being $2\cdot|E|$, is also even. So k is necessarily even. If there are $|V_{odd}|$ odd-degree vertices, k is the sum of $|V_{odd}|$ odd numbers. So k is even.

Q 36: Construct two nonisomorphic simple graphs with six vertices with degrees 1, 1, 2, 2, 3, and 3. What is the number of edges in each graph?

Since the sum of the degrees is 12, the number of edges must be 6. Two nonisomorphic graphs are as follows:

Q 37: Show that if G and G^* are isomorphic graphs, the degree of each vertex is preserved under the isomorphism.

Let $G=(V,E)$ and $G^*=(V^*,E^*)$ be isomorphic under the one-to-one mapping ϕ. The only edges incident with $\phi(v)$ are edges joining $\phi(v)$ and $\phi(u_i)$, where $u_i\in N(v)$. So the degree of $f\phi(v)$ is the same as that of v.

Q 38: Show that it is not possible to have a group of seven people such that each person in the group knows exactly three other people in the group.

Reformulating this problem in terms of graphs, we need to show that it is impossible to have a 3-regular graph with seven vertices. This is indeed impossible, because every k-regular graph, where k is odd, must have an even number of vertices.

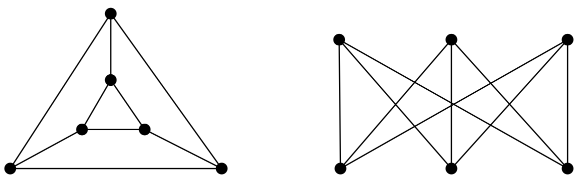
Q 39: Prove that in any group of six people, there will be either three people who know one another or three people who do not know one another.

Reformulate this problem into graph theory as follows: show that for any graph G with 6 vertices, K_3 will be a subgraph of G or a subgraph of $\overline{G}$. Consider vertex $v \in V(G)$. We know that $\delta_G(v) \geq 3$, or otherwise $\delta_{\overline{G}}(v) \geq 3$. Assume $\delta_G(v) = 3$ in G. If there is any edge joining two of v's neighbors in G, we have identified K_3 as a subgraph of G. If there is no such edge between any two of these three neighbors, we have identified K_3 as a subgraph in $\overline{G}$.

Q 40: Show that if a bipartite graph $G = (\{V_1, V_2\}, E)$ is regular, then $|V_1| = |V_2|$.

Assume that the degree of each vertex is k. Clearly, $|E(G)| = k \cdot |V_1| = k \cdot |V_2|$. This means that $|V_1| = |V_2|$.

Q 41: Construct two nonisomorphic cubic (i.e., 3-regular) graphs each with six vertices.



Q 42: Find the maximum number of edges in a bipartite graph.

Let $G = (\{V_1, V_2\}, E)$ be a bipartite graph with $|V_1| = m_1$ and $|V_2| = m_2$. The number of edges in G cannot exceed $m_1 \cdot m_2$, which is a maximum when $m_1 = m_2$. So the maximum number of edges is $(n/2)^2$ when G has n vertices.

Q 43: A k-cube is a simple connected graph with 2^k vertices. Each vertex is represented by a k-bit number. Let $d(u, v)$ be defined as the number of positions in which u and v have a different bit. Two vertices u and v are joined if and only if $d(u, v) = 1$. Show that a k-cube is a k-regular bipartite graph, and find the number of edges in a k-cube.

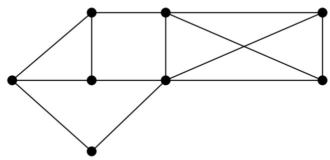
To show that a k-cube is bipartite, we simply construct the set V_1 of vertices represented by the k-bit number of all zeroes (denoted as $\mathbf{0}$), and all k-bit numbers u for which $d(\mathbf{0}, u)$ is even. Likewise, let V_2 consist of all vertices v for which $d(\mathbf{0}, v)$ is odd. Clearly, there can be no edge between vertices from V_1, nor can there be edges between vertices from V_2. We also see that $|V_1| = |V_2|$. With 2^k vertices, this means that $|V_1| = |V_2| = 2^{k-1}$. Each vertex will be joined with k other vertices, i.e., its degree will be k. We conclude that a k-cube has $\frac{1}{2}\sum \delta(v) = (2^k) \cdot k/2$ edges.

Q 44: Find the fewest vertices needed to construct a complete graph with at least 1000 edges.

If the number of vertices is n, we have the inequality $n(n-1)/2 \geq 1000$. Therefore, $n \geq 46$.

Q 45: Test whether $[5, 4, 3, 3, 3, 3, 3, 2]$ is graphic. If it is graphic, draw a simple graph with this sequence as the degree sequence.

1. $v = [5, 4, 3, 3, 3, 3, 3, 2]$ *and* $v_1 = [3, 3, 2, 2, 2, 2, 2]$
2. $v = [3, 3, 2, 2, 2, 2, 2]$ *and* $v_1 = [2, 2, 2, 2, 1, 1]$
3. $v = [2, 2, 2, 2, 1, 1]$ *and* $v_1 = [2, 1, 1, 1, 1]$
4. $v = [2, 1, 1, 1, 1]$ *and* $v_1 = [1, 1, 0, 0]$
5. $v = [1, 1, 0, 0]$ *and* $v_1 = [0, 0, 0]$



Q 46: Test whether $[6, 6, 5, 4, 3, 3, 1]$ is graphic.

1. $v = [6, 6, 5, 4, 3, 3, 1]$ *and* $v_1 = [5, 4, 3, 2, 2, 0]$
2. $v = [5, 4, 3, 2, 2, 0]$ *and* $v_1 = [3, 2, 1, 1, -1]$ *Since we obtain a sequence with a negative element, we conclude that the given sequence is not graphic.*

Q 47: Find the complements of K_n and $K_{m,n}$.

Obviously, the complement of K_n is the empty graph on n vertices, i.e., $E(\overline{K_n}) = \emptyset$. The complement of $K_{m,n}$ consists of two disjoint subgraphs: K_m and K_n.

Q 48: Show that if every edge in a graph joins an odd-degree vertex and an even-degree vertex, the graph is bipartite. Is the converse true?

Let V_1 consist of all odd-degree vertices and V_2 of all even-degree vertices. Clearly, there is no edge between any two vertices from V_1 or from V_2, hence the graph is bipartite. The converse is obviously not true: simply consider $K_{3,3}$.

Q 49: Show that every subgraph of a bipartite graph is also bipartite.

Let $G = (\{V_1, V_2\}, E)$ be a bipartite graph and $H \subseteq G$. Clearly, $V(H) \subseteq V(G)$, in particular, we can partition $V(H)$ into the sets $V(H) \cap V_1$ and $V(H) \cap V_2$. H

will certainly not contain an edge that joins any two vertices belonging to the same subset.

Q 50: Prove that for any graph G, $\kappa(G) \leq \lambda(G) \leq \min\{\delta(v)|v \in V(G)\}$

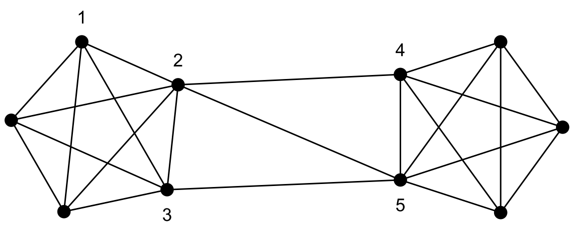
That $\lambda(G) \leq \min\{\delta(v)|v \in V(G)\}$ is easy to see. Consider a vertex u with minimal degree, that is, $\delta(u) = \min\{\delta(v)|v \in V(G)\}$. If we simply remove the $\delta(u)$ edges incident with u, then u will become isolated, and certainly the resulting graph will have at least one more component then it had before (namely the one consisting only of u).

To prove that $\kappa(G) \leq \lambda(G)$, consider a graph G with $\lambda(G) = k$ and let $E^* = \{e_1, e_2, \ldots, e_k\}$ be a minimal edge cut of G, with $e_i = \langle u_i, v_i \rangle$. Let U denote the set of vertices $\{u_1, \ldots, u_k\}$ and V the set $\{v_1, \ldots, v_k\}$. Note that in this case, the vertices in either set need not be distinct. The graph $G - E^*$ will fall apart into exactly two components, say G_1 and G_2. If G_1 contains a vertex u distinct from any u_i then clearly removing all vertices in U will disconnect u from any vertex in G_2, so that $\kappa(G) \leq k$.

If there is no such vertex u, then assume that $V(G_1) = U$. Consider vertex u_1. We know that u_1 is adjacent to d_1 vertices from G_1, and each of these neighbors in G_1 is adjacent to a vertex from V. Let E_1^* be a set of edges from E^* joining vertices from the d_1 neighbors of u_1 and exactly one vertex from V. Likewise, let E_2^* be the d_2 edges from E^* incident with u_1. Obviously, $d_1 + d_2 = |E_1^* \cup E_2^*| \leq |E^*|$. Also, the $d_1 + d_2$ neighboring vertices of u_1 form a vertex cut. This also means that $\kappa(G) \leq d_1 + d_2 \leq |E^*| = \lambda(G)$, completing the proof.

Q 51: Construct a graph for which $\kappa(G) < \lambda(G) < \min\{\delta(v)|v \in V(G)\}$ is strict.

Consider the following graph. Clearly, $\delta(1) = 4$ and is also the minimum vertex degree of G. Furthermore, the set $\{\langle 2,4 \rangle, \langle 2,5 \rangle, \langle 3,5 \rangle\}$ forms a minimal edge cut of size 3, whereas the set of vertices $\{2,5\}$ forms a minimal vertex cut of size 2.



Q 52: Provide an algorithm for checking whether an undirected graph G is connected.

Let $R_t(u)$ denote the set of reachable vertices from u found after t steps. Further-

more, let $N(v)$ denote the set of neighbors of v, that is, $N(v) = \{w \in V(G) | \exists \langle v, w \rangle \in E(G)\}$.

1. *Set $t \leftarrow 0$ and $R_0(u) \leftarrow \{u\}$.*
2. *Construct the set $R_{t+1}(u) \leftarrow R_t(u) \cup_{v \in R_t(u)} N(v)$.*
3. *If $R_{t+1}(u) = R_t(u)$, stop: $R(u) \leftarrow R_t(u)$. Otherwise, increment t and repeat the previous step.*

Q 53: Prove that the Harary graph $H_{k,n}$ is k-connected.

Let us first consider the case that k is even. Our proof is completed if we can show that there is no vertex cut with fewer than k vertices. To this end, let us assume that such a set W does exist. If we can then prove that this assumption can never hold, we will have completed our proof (we come back to this method of proving a theorem below).
To this end, let vertices i and j belong to different components of $H_{k,n} - W$ (i.e., $G[V(H_{k,n}) \backslash W]$). Consider the set $N_{i \to j}$ of left-hand neighbors of i, including i: $\{i, i+1, \ldots, j-1, j\}$, and likewise its right-hand neighbors $N_{i \leftarrow j} = \{j, j+1, \ldots, i-1, i\}$. In both cases, addition is taken modulo n. Let $W_{i \to j} \stackrel{\text{def}}{=} W \cap N_{i \to j}$ and $W_{i \leftarrow j} \stackrel{\text{def}}{=} W \cap N_{i \leftarrow j}$ (meaning that $W = W_{i \to j} \cup W_{i \leftarrow j}$). We know that $|W| < k$, so we must have that either $|W_{i \to j}| < k/2$ or $|W_{i \leftarrow j}| < k/2$. Assume that $|W_{i \to j}| < k/2$
Now consider an arbitrary vertex u in $H_{k,n} - W$, lying on, say, segment S_1. We know that u is adjacent to $k/2$ consecutive vertices in either direction. As a consequence, removing less than $k/2$ vertices as is done through $W_{i \to j}$ will still allow us to reach any vertex v on segment S_2. In other words, $H_{k,n} - W$ will remain connected, contradicting our assumption that W was a vertex cut.

Q 54: Prove that for a connected acyclic simple graph G with n vertices, $|E(G)| = n - 1$.

We prove this lemma by induction on the number of vertices. Clearly, when $n = 1$ there can be no edges and the lemma is seen to hold. Now assume the lemma holds for all acyclic simple graphs with less than n vertices. Let H be an acyclic simple graph with $n \geq 2$ vertices, and edge $\langle u, v \rangle \in E(H)$. If we remove this edge, then the result will be two separate subgraphs G_1 and G_2, for otherwise $\langle u, v \rangle$ was part of a cycle. Both subgraphs are acyclic, each with less than n vertices, so that $|E(G_1)| = |V(G_1)| - 1$ and $|E(G_2)| = |V(G_2)| - 1$. Because we have not removed any vertices, we know that

$$|E(H)| = |E(G_1)| + |E(G_2)| + 1 = |V(G_1)| - 1 + |V(G_2)| - 1 + 1 = n - 1$$

which completes the proof.

Q 55: Prove that for a plane graph G with n vertices, m edges, and r regions, we have that $n - m + r = 2$.

The proof is by induction on r, the number of regions. If $r = 1$, then there is only a single region, which means there cannot be a region enclosed by edges of G. In other words, G must be acyclic, in which case $m = n - 1$ and thus $n - m + r = n - (n-1) + 1 = 2$. For $r = 1$ the formula is therefore seen to be true.
Now assume the formula is true for all plane graphs with less than r regions, and let G be a plane graph with $r > 1$ regions. Choose an edge e (which is not a cut edge) and consider the subgraph $G^* = G - e$. As e was part of a cycle, we will have merged two regions, reducing the total number of regions by 1. In that case, we know that Euler's formula is true, and as a consequence, $|V(G^*)| - |E(G^*)| + (r-1) = 2$. Considering that $|V(G^*)| = |V(G)|$ and $|E(G^*)| = |E(G)| - 1$, we now obtain $|V(G)| - (|E(G)| - 1) + r - 1 = |V(G)| - |E(G)| + r = 2$, completing our proof.

Q 56: Prove that for any connected simple planar graph G with $n \geq 3$ vertices and m edges, we have that $m \leq 3n - 6$

Consider a region f in any plane graph of G. For any interior region, let $B(f)$ denote the number of edges by which f is enclosed, i.e., the length of its "border." Obviously, $B(f) \geq 3$ for any interior region. However, with $n \geq 3$ we also have that the exterior region is "bounded" by at least 3 edges. Therefore, if there are a total of r regions, then clearly $\sum B(f) \geq 3r$. On the other hand, it is not difficult to see that $\sum B(f)$ counts every edge in G once or twice, and hence $\sum B(f) \leq 2m$, so that we obtain $3r \leq \sum B(f) \leq 2m$, and thus $r \leq \frac{2}{3}m$. From Euler's formula we then derive that $m = n + r - 2 \leq n + \frac{2}{3}m - 2$, so that $m \leq 3n - 6$.

Q 57: Show that K_5 is nonplanar.

With $n = |V(K_5)| = 5$ and $m = |E(K_5)| = \binom{5}{2} = 10$, we have that $m \not\leq 3n - 6$, so that K_5 cannot be planar.

Q 58: Show that the complete bipartite graph $K_{3,3}$ is nonplanar.

Each interior region f in any $K_{p,q}$ will necessarily be enclosed by an even number of edges. If $B(f)$ denotes the number of edges enclosing interior region f, and realizing that also the exterior region will be "bounded" by at least four edges, we find that $\sum B(f) \geq 4r$, where r is the total number of regions. Because edges are counted twice, we should have that $4r \leq 2m = 18$. However, Euler's formula tells us that $r = 2 - n + m = 2 - 6 + 9 = 5$, so that $4r \not\leq 18$. Therefore, $K_{3,3}$ cannot be planar.

Problems Chapter 3

Q 59: Show that for any simple undirected graph with m egdes there are 2^m possible orientations. What can we say about the number of orientations for nonsimple graphs?

Every edge can be ordered in two possible ways. Because the graph is simple, we know there are no loops nor parallel edges. Hence, there are 2^m different combinations of ordering the edges, and thus 2^m different orientations. When dealing with a nonsimple graph, the situation is more complicated. First, each loop will have only one distinguishable orientation. If we have, say, two edges $e_1 = \langle u,v \rangle$ and $e_2 = \langle u,v \rangle$, then, in principle, we could have four different orientations:

- *O_1 has arcs $\{a_1 = \langle \overrightarrow{u,v} \rangle, a_2 = \langle \overrightarrow{u,v} \rangle\}$*
- *O_2 has arcs $\{a_1 = \langle \overrightarrow{u,v} \rangle, a_2 = \langle \overrightarrow{v,u} \rangle\}$*
- *O_3 has arcs $\{a_1 = \langle \overrightarrow{v,u} \rangle, a_2 = \langle \overrightarrow{u,v} \rangle\}$*
- *O_4 has arcs $\{a_1 = \langle \overrightarrow{v,u} \rangle, a_2 = \langle \overrightarrow{v,u} \rangle\}$*

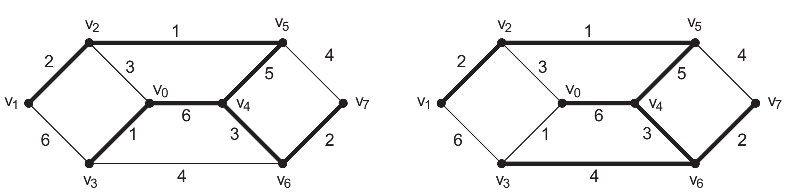
Arguably, orientation O_2 and O_3 can be considered identical. Note that when edges have weights, we would need to take this into account when considering whether or not two orientations are different.

Q 60: In Dijkstra's algorithm, we set $R_t(u) = S_t(u) \cup_{v \in S_t(u)} N(v)$, and later consider vertices from $R_t(u) \backslash S_t(u)$. Why can't we directly consider the set $\cup_{v \in S_t(u)} N(v)$?

The answer is quite simple when realizing that a set $N(v)$ may also contain vertices from $S_t(u)$. In other words, $R_t(u) \backslash S_t(u) \neq \cup_{v \in S_t(u)} N(v)$. For Dijkstra's algorithm, it is important to consider only vertices outside the set $S_t(u)$.

Q 61: Apply Dijkstra's algorithm for vertex v_4 from Figure 3.4 and compute the weight of the resulting rooted tree $T(v_4)$. Find an alternative tree $T^*(v_4)$ that also gives shortest paths originating from v_4, but with a different weight, that is, $w(T(v_4)) \neq w(T^*(v_4))$.

The following figure shows two trees rooted at v_4, both containing shortest paths from v_4 to any other vertex. However, their weights are different, as can be readily observed.



Q 62: Change Dijkstra's algorithm so that it can be applied to weighted, strongly connected *directed* graphs.

Perhaps not surprisingly, Dijkstra's algorithm will also work for strongly connected digraphs, provided that we consider $N(v)$ to be the set of vertices w for which there is an arc $\langle \overrightarrow{v,w} \rangle$. The algorithm is otherwise unaltered.

Q 63: Let G be an undirected graph and $\mathcal{E} \stackrel{\text{def}}{=} \{E_1, \ldots, E_k\}$ a partitioning of its edge set. Let V_i be the collection of end points of edges from E_i. Prove that $\mathcal{E}$ is an edge coloring if and only if $|V_i| = 2 \cdot |E_i|$.

Clearly, if $|V_i| = 2 \cdot |E_i|$, then this can mean only that all edges in E_i have different end points, and thus that no two edges have an end point in common. By definition, $\mathcal{E}$ is an edge coloring of G.

Likewise, if $\mathcal{E}$ is an edge coloring, no two edges in any E_i will share a vertex, meaning that necessarily $|V_i| = 2 \cdot |E_i|$.

Q 64: A manufacturer of chemical goods is faced with the problem that certain goods cannot be stored at the same place due to the danger of unwanted reactions. What he seeks is a storage scheme such that goods that cannot be located at the same place are indeed separated. Provide a graph-theoretical model to solve this problem.

Every good is modeled as a vertex. If two goods cannot be stored at the same location, we join their associated vertices. What we are then seeking is a vertex coloring of the resulting graph.

Q 65: Design a simple algorithm by which we can identify the components of a graph.

Consider a graph G. Start with an arbitrary vertex $v \in V(G)$ and set the current label $l \leftarrow 1$.

1. *Set V_l equal to $\{v\}$.*
2. *Execute Algorithm 3.1 to find all vertices $R(v)$ that can be reached from v and add each vertex in $R(v)$ to V_l.*
3. *Increment l: $l \leftarrow l + 1$, and choose an arbitrary vertex v again, but now from the set of vertices not yet inspected: $V(G) \setminus \cup_{i=1}^{l-1} V_i$. If no such vertex exists, stop. Otherwise, continue with the first step.*

The effect is that all vertices belonging to the same component will have the same label.

Q 66: Prove that there exists an orientation $D(G)$ for a connected undirected graph G that is strongly connected if and only if $\lambda(G) \geq 2$. In other words, G cannot be 1-edge-connected.

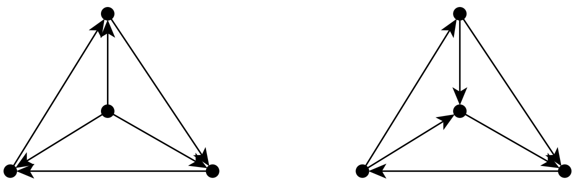
Let us first consider a strongly connected orientation D of G. We prove, by contradiction, that G is 2-edge-connected. To that end, assume that G is not 2-edge-connected and that the removal of $e = \langle u, v \rangle$ disconnects G, that is $G - e$ falls into two components G_1 and G_2. Clearly, we can assign only one orientation to e, that is, $D(G)$ will either contain the arc $a = \langle \overrightarrow{u, v} \rangle$ or the arc $a' = \langle \overrightarrow{v, u} \rangle$. Because all paths in G from a vertex $x \in V(G_1)$ to a vertex $y \in V(G_2)$ will contain e, it is also clear that with either orientation of e, $D(G)$ cannot be strongly connected, which violates our initial assumption. Hence, G cannot be 1-edge-connected and therefore is (at least) 2-edge-connected.

Now consider a 2-edge-connected undirected graph G. We construct an orientation D of G that is strongly connected. We know that every edge of G lies on a cycle. Consider the cycle $C = [v_1, v_2, \ldots, v_n, v_1]$. We replace each edge $\langle v_i, v_{i+1} \rangle$ with an arc $\langle \overrightarrow{v_i, v_{i+1}} \rangle$ and edge $\langle v_n, v_0 \rangle$ with arc $\langle \overrightarrow{v_n, v_1} \rangle$. Any edge $\langle v_i, v_j \rangle$ between nonadjacent vertices on C can be oriented arbitrarily. This situation is shown in (a) below. Clearly, if $V(C) = V(G)$ we will have constructed a strongly connected orientation of G.

Assume $V(C) \neq V(G)$ so that we have not yet covered all vertices of G. Let w be such a vertex, i.e., $w \notin V(C)$. Because G is 2-edge-connected, we know that there are two edge-independent paths connecting w to v_1, as shown (b) above. Without loss of generality, we may assume that these two paths partly overlap with C. One path, P_1, will have the form $[w = w_1, w_2, \ldots, w_k, v_j, v_{j+1}, \ldots, v_1]$. The other will necessarily have the form $[w = \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_l, v_i, v_{i-1}, \ldots, v_1]$, where $1 \leq i \leq j \leq n$. We then transform each edge $\langle w_x, w_{x+1} \rangle$ to the arc $\langle \overrightarrow{w_x, w_{x+1}} \rangle$, and each edge $\langle \overline{w}_y, \overline{w}_{y+1} \rangle$ to $\langle \overrightarrow{\overline{w}_{y+1}, \overline{w}_y} \rangle$. Again, edges between nonadjacent vertices on P_1 and P_2 may be oriented arbitrarily. It should be clear that all vertices in $W = V(C) \cup V(P_1) \cup V(P_2)$ are connected through two edge-disjoint paths in D.

If there is still a vertex in $V(G) \backslash W$, we simply repeat the procedure until all edges have been provided with an orientation. The result will be a strongly connected orientation of G.

Q 67: Any orientation of the complete graph with vertex set $\{1,2,\ldots,n\}$ is a tournament. A tournament is **transitive** if there is an arc from i to k whenever there is an arc from i to j and an arc from j to k for each i, j, and k. Construct both a transitive tournament with four vertices and one that is not transitive.



Q 68: Prove that in a digraph, the sum of the outdegrees of all the vertices is equal to the number of arcs, which is also equal to the sum of the indegrees of all the vertices.

The outdegree of a vertex v is the number of arcs having v as their tail. So when we add all the outderees, each arc is counted exactly once. Likewise, when the indegrees are summed, each arc is counted exactly once. Thus the sum of the outdegrees and the sum of the indegrees are both equal to the total number of arcs in the digraph.

Q 69: Prove that every walk in a graph between vertices v and w contains a path between v and w, and every directed walk from v to w in a digraph contains a directed path from v to w.

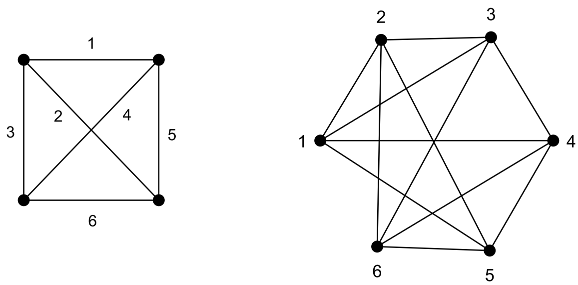
Let W be a walk between v and w. If $v = w$, there is the trivial path with no edges. Therefore, assume that v and w are not the same vertex. Suppose W is the walk $[v = v_0, v_1, \ldots, v_n = w]$. It is possible that the same vertex occurs more than once, that is, it may be that $v_i = v_j$. If no vertex of the graph appears more than once in the sequence, we have a path between v and w. Otherwise, there will be at least one vertex that appears as v_i and v_j in the sequence with $i < j$. If we remove the terms $v_{i+1}, v_{i+2}, ..., v_j$ from the sequence, we still have a walk between v and w that contains fewer edges. We continue this process until each repeated vertex appears only once in the walk; at that stage, we have a path between v and w. The proof in the case of directed walks is similar.

Q 70: Show that a graph G is bipartite if and only if $\chi(G) = 2$

In the bipartite graph $G = (\{V_1, V_2\}, E)$, assign the same color (say red) to each vertex in V_1. Then assign a unique color other than red (say blue) to each vertex in V_2. Thus the chromatic number of G is 2. On the other hand, suppose the chromatic number of $G = (V, E)$ is two. Let V_1 be the set of vertices such that each vertex in V_1 has the same color. Let $V_2 = (V \backslash V_1)$. Then every edge in G is between a vertex in V_1 and a vertex in V_2, meaning that G is bipartite.

Q 71: Construct the line graph of K_4.

The following graph shows the relationship between K_4 and its line graph.



Q 72: If v is the vertex in the line graph $L(G)$ that corresponds to the edge joining vertex x and vertex y in G, find the degree of v in $L(G)$.

$\delta_{L(G)}(v) = \delta_G(x) + \delta_G(y) - 2$

Q 73: How many vertices does $L(K_n)$ have? And what about the number of edges?

It should be clear that $|V(L(K_n))| = |E(K_n)| = \binom{n}{2}$. We can compute the number of edges by taking a look at the vertex degrees in $L(K_n)$. An edge joining two vertices u and v in K_n is incident with $\delta(u) + \delta(v) - 2 = (n-1) + (n-1) - 2 = 2(n-2)$ other edges. In other words, the degree of a vertex in $L(K_n)$ is equal to $2(n-2)$. Hence, the total number of edges in $L(K_n)$ is $\frac{1}{2}\sum \delta(v) = \frac{1}{2}n(n-1)(n-2)$.

Q 74: Let G be a simple graph with n vertices. Compute the number of edges in $L(G)$.

We know that each edge $e = \langle u, v \rangle$ in G, we'll have a vertex in $L(G)$. In particular, $\delta_{L(G)}(e) = \delta(u) + \delta(v) - 2$. If we consider all edges in G, it is not difficult to see that wehn computing the vertex degree of those edges in $L(G)$, an end point u will be counted exactly $\delta(u)$ times in the summation $\sum_{w \in L(G)} \delta(w)$. With n vertices in $L(G)$, this means that

$$\sum_{e=\langle u,v \rangle} \delta_{L(G)}(e) = \sum \big(\delta_G(u) + \delta_G(v) - 2\big) = \sum \big(\delta_G(u)\big)^2 - 2 \cdot n$$

We conclude that $|E(L(G))| = -n + \frac{1}{2}\sum_{u \in V(G)} \big(\delta_G(u)\big)^2$.

Q 75: Suppose G is a simple graph with five vertices with degrees 1, 2, 3, 3, and 3. Find the number of vertices and edges in $L(G)$.

The sum of degrees is 12, so G has six edges. Thus $L(G)$ has six vertices. The sum of the squares of the degrees is 32. Hence, the number of edges in $L(G)$ is $(32-12)/2 = 10$.

Q 76: Show that there is no graph G such that $L(G) = K_{1,3}$.

Since $K_{1,3} = (\{V_1, V_2\}, E)$ has four vertices, if there is a graph G it should have four edges. Suppose these four edges are a, b, c, and d, and assume that $V_1 = \{a\}$ (and $V_2 = \{b, c, d\}$). For G, we would require that edge a has an end point that is also an end point of b, c and d, while at the same time no two of these three edges are allowed to have an end point in common.

Q 77: Construct an example to show that if $L(G)$ and $L(H)$ are isomorphic, it is not necessary that G and H are isomorphic.

It is easy to see that $L(K_3) = K_3 = L(K_{1,3})$, yet obviously, K_3 and $K_{1,3}$ are not isomorphic.

Q 78: Show that

(a) a graph G is isomorphic to its line graph if and only if the degree of each vertex is 2

(b) the line graph of a connected graph G is (isomorphic to) K_n, if and only if G is (isomorphic to) $K_{1,n}$, when $n > 3$.

(a) If the degree of each vertex of G is 2, the degree of each vertex of $L(G)$ is also 2, and G and $L(G)$ both have the same number of vertices and the same number edges. So G and $L(G)$ are isomorphic. Conversely, if both G and $L(G)$ are isomorphic, they both have the same number of vertices and same number of edges, and the degree of each vertex is 2.

(b) If $n > 3$, $L(K_{1,n}) = K_n$. Conversely, if $L(G) = K_n$, G has n edges, and all these edges have exactly one vertex in common since the degree of each vertex in K_n is $(n-1)$.

Q 79: Show that a digraph $D = (V, A)$ is strongly connected if and only if for every nonempty subset $X \subseteq V$ there exists an arc $\langle \overrightarrow{x, y} \rangle$ from a vertex $x \in X$ to a vertex $y \in V \backslash X$.

Suppose $D = (V, A)$ is strongly connected and X is an arbitrary nonempty subset of V. Let u be an arbitrary vertex in X and y be an arbitrary vertex in $Y = V \backslash X$. Then there is at least one directed path P from u to v that will have an arc $\langle \overrightarrow{x, y} \rangle$, where $x \in X$ and $y \in Y$.

Conversely, assume that D is a digraph that satisfies the property. Suppose D is not strongly connected and u and v are two vertices such that there is no directed path from u to v in D. Let X be the set of all vertices that are terminal vertices of directed paths that originate from u. By assumption, X is a proper subset of V. So there exists an arc e from vertex $x \in X$ to vertex $y \in V \backslash X$. Now x is the terminal vertex of a directed path P from u, and this path can be enlarged into path P^* from u to y using arc e. Thus the vertex y cannot be in $V \backslash X$, which is a contradiction.

Q 80: Show that a vertex v of a connected graph is a cut vertex if and only if there exist two distinct vertices u and w such that every path between these two vertices passes through v.

Let v be a cut vertex in a connected graph G. Then $G - v$ has at least two components. If we choose u from one component and w from another component, any path in G between u and w has to pass through v. On the other hand, suppose there are two vertices in a connected graph such that every path between these two vertices passes through vertex v. If this vertex is deleted, there cannot be a path between these two vertices in the resulting graph. In other words, this deletion makes G disconnected. So v is a cut vertex of G.

Q 81: Show that any nontrivial graph has at least two vertices that are not cut vertices.

We may assume without loss of generality that graph G is connected. Let P be a shortest path of maximum length in G, with end points u and v. Suppose u is a cut vertex. Then $G - u$ has at least two components. Let w be a vertex in a component that does not contain v. In G, there is a path between v and w. Since u is a cut vertex, this path has to pass through u, implying that $d(v, w) > d(u, v)$, which contradicts the maximality of $d(u, v)$. So u is not a cut vertex. Similarly, v is also not a cut vertex.

Q 82: Show that an edge of a connected graph is a cut edge if and only if there exist vertices v and w such that every path between these two vertices contains this edge.

The deletion of a cut edge from a connected graph creates two connected components of the graph, and any path in the original graph joining a vertex in one component and a vertex in the other component contains the cut edge. On the other hand, if there are two vertices such that every path between these two vertices contains the same edge, the deletion of this edge will disconnect the graph.

Q 83: Show that an edge is a cut edge if and only if no cycle contains that edge.

Assume without loss of generality that the graph under consideration is connected. If e is a cut edge joining two vertices u and v and if there is a cycle that contains

this edge, there is a path in the graph between these two vertices other than the edge. Since the graph is connected, every vertex in the graph is connected to every vertex in the cycle. So the deletion of e will not disconnect the graph, which contradicts that e is a cut edge.
Conversely, let G be a connected graph and e be an edge joining u and v such that no cycle contains this edge. Suppose e is not a bridge. Then $G - e$ is still a connected graph that has a path joining u and v. This path and the edge e together constitute a cycle containing e in G, contradicting the hypothesis.

Q 84: Show that in a graph with n vertices, the length of a path cannot exceed $(n-1)$ and the length of a cycle cannot exceed n.

If u and v are two vertices, the path between u and v can have at most $(n-2)$ distinct vertices. So the maximum length of the path is $(n-1)$. Likewise, the maximum length of a cycle is n.

Q 85: Show that if a simple graph G with n vertices and m edges has k components, $m \leq \frac{1}{2}(n-k)(n-k+1)$.

(a) The conclusion remains valid even if we assume that each component is a complete graph. Suppose H_i, and H_j are two such components with n_i and n_j vertices, where $n_i \geq n_j \geq 1$. If we replace these two components by two complete graphs of (n_i+1) and (n_j-1) vertices, respectively, the total number of vertices will remain unchanged but the number of edges will increase by $n_i - n_j + 1$. So the number of edges of a simple graph of with n vertices and k components will be a maximum if there are $(k-1)$ isolated vertices and one component that is a complete graph with $(n-k+1)$ vertices with $\frac{1}{2}(n-k)(n-k+1)$ edges.

Q 86: Find the minimum number of edges in a k-connected graph.

If the graph G with n vertices and m edges is k-connected, the degree of each vertex is at least k and so $2 \cdot m$ is at least $n \cdot k$.

Q 87: Draw a k-connected graph with n vertices and m edges such that $2 \cdot m = n \cdot k$ when (a) $k = 1$ and (b) $k = 2$.

(a) K_2

(b) $G = (V, E)$ *with* $V = \{1, 2, 3, 4\}$ *and edges* $\langle 1, 2\rangle$, $\langle 2, 3\rangle$, $\langle 3, 4\rangle$, *and* $\langle 4, 1\rangle$.

Q 88: Prove that a graph G is bipartite if and only if it contains no cycles of odd length.

First assume G is bipartite with its vertex set V partitioned into subsets V_1 and V_2. Let $C = [v_1, v_2, \ldots, v_n = v_1]$ be a cycle of length n. Assume that $v_1 \in V_1$. Clearly, we must have that $v_2 \in V_2$, and thus that $v_3 \in V_1$, and so on. In other

words, each vertex $v_{2i-1} \in V_1$ and $v_{2i} \in V_2$, for $1 \leq i \leq n/2$. Furthermore, we know that $v_n = v_1$, meaning that $v_{n-1} \in V_2$. This is possible only if $n - 1 = 2j$ for some j, meaning that n is odd.
Without loss of generality, assume G is connected. Assume that G contains no cycle of odd length. Let u be an arbitrarily chosen vertex vertex. Let $R_k(u)$ be the set of vertices reachable from u through a path of length k. If there is an i and $j \neq i$ for which $R_{2i}(u) \cap R_{2j+1}(u)$ is not empty, we have found a vertex v that is reachable from u by means of path of even length, and another path of odd length, meaning that we would have found an odd-length cycle. We conclude that the sets $\bigcup R_{2i}(u)$ and $\bigcup R_{2j+1}(u)$ are disjoint, meaning that G is bipartite.

Q 89: Show that the minimum number of time slots needed for the class-scheduling problem is the value of $\chi(G)$ of the associated graph G.

We first prove that we need at most $\chi(G)$ slots to schedule all classes. From the definition of chromatic number, we know that any two vertices with the same color cannot be adjacent. This also means that the two classes associated with those two vertices need not be taken by the same group of students. Hence, they can be scheduled at the same time, that is, for the same time slot. In general, all vertices with the same color represent the set of classes that can be scheduled at the same time. This means that $\chi(G)$ slots are sufficient to schedule all classes.
We now prove that we need at least $\chi(G)$ slots to schedule all classes. Suppose that $k < \chi(G)$ slots are sufficient. Classes in the same slot should be taught to different groups. In the graph G, this means that the vertices representing those classes should be nonadjacent. As a consequence, we should be able to use only k different colors yielding a k-vertex coloring of G, which contradicts the fact that $\chi(G)$ is minimal.

Q 90: Show that for any (simple, connected) graph G, $\chi(G) \leq \Delta(G) + 1$.

We prove that the theorem holds by induction on the number n of vertices of G. For $n = 1$, we need to consider the complete graph K_1. Obviously, $\chi(K_1) = 1$ and $\Delta(K_1) = 0$, so that the theorem holds.
Now assume the theorem holds for all graphs on $k > 1$ vertices, and consider a graph G with $k + 1$ vertices. Let vertex $v \in V(G)$ with $\delta(v) = \Delta(G)$. The graph $G^* = G - v$ has k vertices, so there exists a vertex coloring C^* of G^* with $\chi(G^*) \leq \Delta(G^*) + 1$ different colors. If $\Delta(G) = \Delta(G^*)$, then in the worst case, the number of colors used in G^* is $\chi(G^*) = \Delta(G^*) + 1 = \Delta(G) + 1$. Considering that v has $\Delta(G) - 1$ neighbors, this means that there is a color available from the ones used in G^* that we can use for v and which has not been used for any of v's neighbors.
On the other hand, if $\Delta(G) > \Delta(G^*)$, then we can simply permit ourselves to introduce a new color for v and use the ones from an optimal coloring of G^* for all other vertices. At worst, we will then have that $\chi(G) = \chi(G^*) + 1 \leq \Delta(G^*) + 2$.

If $\Delta(G^*) < \Delta(G)$, then the smallest value of $\Delta(G)$ for which this inequality is true, is, of course, when $\Delta(G) = \Delta(G^*) + 1$. Therefore, we know that $\Delta(G^*) + 2 \leq \Delta(G) + 1$, so that we indeed have that $\chi(G) \leq \Delta(G) + 1$.

Q 91: Show that every planar graph G has a vertex v with $\delta(v) \leq 5$.

For all planar graphs with $n \leq 6$ vertices, the theorem is obviously true. For planar graphs with $n > 6$, we prove the theorem by contradiction. To this end, consider a planar graph G for which $n > 6$. Let m be the number of edges of G. We know that $\sum_{v \in V(G)} \delta(v) = 2m$. Therefore, if there is no vertex with degree 5 or less, then $6n \leq 2m$. In addition, we know that $m \leq 3n - 6$, and thus that $6n \leq 6n - 12$. Obviously, this is false, meaning that our assumption that there is no vertex with degree 5 or less must be false as well.

Q 92: Show that for any planar graph G, $\chi(G) \leq 5$.

Let $n = |V(G)|$. For $n = 1$, the theorem is obviously true. Assume the theorem holds for all planar graphs with $k > 1$ vertices and consider a graph G with $k + 1$ vertices. Let vertex v with $\delta(v) \leq 5$ (we just proved that such a vertex exists), and consider the graph $G^* = G - v$. Because $|V(G^*)| = k$, we know there exists a 5-vertex coloring of G^*, with, say, colors $c_1, \ldots, c_5$. If not all of these colors are used by the vertices in the neighbor set $N(v)$ of v, we can assign the unused color to v and will thus have constructed a 5-vertex coloring of G.
Consider the situation that all five colors have been used for coloring the vertices of $N(v)$. Note that $\delta(v) = 5$ so that we may assume that $N(v) = \{v_1, \ldots, v_5\}$ and that vertex v_i has color c_i according to a clockwise ordering of these vertices around v. We will rearrange the colors of G^* such that we can assign one of the colors c_i to v.
Let us first assume that there is no (v_1, v_3)-path in G^* for which all vertices have been colored either c_1 or c_3. Now consider all paths in G^* that originate in v_1 and for which the vertices are colored either c_1 or c_3. These paths induce a subgraph H of G^*. Note that $v_3 \notin V(H)$, as this would mean that there is a (v_1, v_3)-path. For the same reason, none of v_3's neighbors can be in H, i.e., $N(v_3) \cap V(H) = \emptyset$. What we can then do is interchange the colors c_1 and c_3 in H, which leads to another 5-vertex coloring of G^*. However, in this case, vertex v_1 will be colored c_3, and none of the vertices in $N(v)$ will be colored c_1. Therefore, we can use c_1 for v.
Let us now assume that there is a (v_1, v_3)-path P in G^* for which all vertices have been colored either c_1 or c_3. Consider the cycle $[v_3, v, v_1, P]$. This cycle either encloses v_2, or it encloses v_4 and v_5. Hence, because G is planar, there can be no (v_2, v_4)-path in G^* whose vertices are colored using only c_2 and c_4. Again, consider all paths originating in v_2 and that have either color c_2 or c_4. As before, these paths induce a subgraph H' of G^*. We interchange the colors of the vertices in H', allowing us to assign color c_2 to v, and thus leading to a 5-vertex coloring of G.

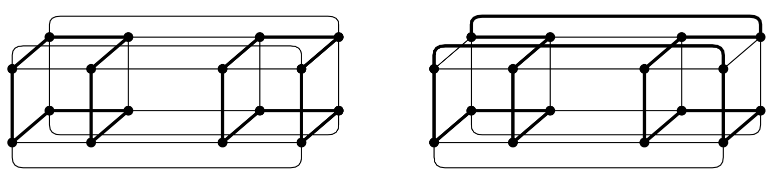
Problems Chapter 4

Q 93: Consider a connected weighted graph G with two vertices of odd degree: u and v. Prove that by duplicating every edge on a minimum-weight (u, v)-path, we obtain a minimum-weight Eulerian graph.

First note that by duplicating edges of any (u, v)-path P will transform G to a Eulerian graph. This can be easily seen when realizing that both u and v will become incident with one extra edge, and that every intermediate vertex on path P will become incident with two extra edges. As a consequence, all vertices in the transformed graph will have even degree, meaning that the graph is Eulerian.
Let G^* be the transformed graph obtained by duplicating the edges of a minimum-weight (u, v)-path. Let G^{**} be a graph resulting from taking any (u, v)-path and duplicating its edges. It is not difficult to see that $w(G^*) \stackrel{\text{def}}{=} \sum_{e \in E(G^*)} w(e) \leq \sum_{e \in E(G^{**})} w(e) \stackrel{\text{def}}{=} w(G^{**})$, which completes our proof.

Q 94: A k-cube is a simple connected graph with 2^k vertices. Each vertex is represented by a k-bit number. Let $d(u, v)$ be defined as the number of positions in which u and v have a different bit. Two vertices u and v are joined if and only if $d(u, v) = 1$. Show that a k-cube is Hamiltonian.

The proof is easy, and can be done by induction. If $k = 1$, we simply need to visit each vertex of a two-vertex graph with an edge connecting them.
Assume that the statement is true for $k > 1$. To build a $(k + 1)$-cube, we take two copies of the k-cube and connect the corresponding edges. Take the Hamiltonian cycle on one cube and reverse it on the other. Then choose an edge on one that is part of the cycle and the corresponding edge on the other and delete them from the cycle. Finally, add to the path connections from the corresponding end points on the cubes which will produce a ccycle on the $(k + 1)$-cube:



Q 95: Show that a graph is Eulerian if and only if it is connected and if the set of its edges can be partitioned into a disjoint union of cycles.

Suppose G is an Eulerian graph. Then each vertex in G has an even degree of at least 2. Thus there is at least one cycle C_1 in the graph. If G is not this cycle, let $G_1 = G - E(C_1)$ Since every vertex in a cycle is of degree 2, every vertex *in G_1 is also even and, as before, has cycle C_2. Let $G_2 = G_1 - E(C_2) = G - E(C_1) - E(C_2)$. We repeat this process of identifying cycles until we get the graph $G_k = G - E(C_1) - E(C_2) - \cdots - E(C_k)$ with no edges. Thus the set of edges is the disjoint union of these k cycles.*

Conversely, suppose the set of edges in a connected graph G is the disjoint union of k cycles. Consider any one of these cycles, say cycle C_1. Since the graph is connected, there is a cycle (say C_2) such that the two cycles have a vertex v, in common. Let Q_{12} be the closed walk that consists of all the edges in these two cycles. As before, there is a cycle C_3 such that this cycle and walk Q_{12} have no edge in common but do have vertex v_2 in common. Let Q_{123} be the closed walk that contains all the edges of these three edge-disjoint cycles. We repeat this procedure until we get a closed walk that contains all the edges of the graph. Thus the graph is Eulerian.

Q 96: Prove that a connected graph G (with more than one vertex) has an Euler tour if and only if it has no vertices of odd degree.

First, assume that P is an Euler tour of G, originating and ending in, say, vertex v. Consider a vertex u different from v. Obviously, u lies on P and for each edge $\langle w_1, u\rangle \in E(P)$ that is used for "entering" u, there is a unique other edge $\langle u, w_2\rangle$ traversed for "leaving" u. Moreover, because these edges are traversed exactly once, edges for entering u are always uniquely paired with edges for leaving u. Hence, the degree of u must be even. By a similar reasoning, the degree of v must also be even. We conclude that all vertices of G have even degree.

Conversely, assume that all vertices of G are of even degree. We now need to prove that G has an Euler tour. To this end, select an arbitrary vertex v and construct a trail P by subsequently traversing edges until it is no longer possible to traverse an edge not belonging to P. Let w be the vertex where P ends. If $w \neq v$, then clearly we have "entered" w once more than we have "left" it, meaning that $\delta(w)$ is odd. This violates our assumption, hence $w = v$ and hence P must be a closed trail.

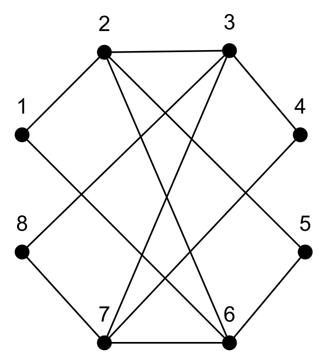
If $E(P) = E(G)$ we have just constructed an Euler tour and we're done. Now assume $E(P) \neq E(G)$, that is $E(P) \subset E(G)$. Because G is connected, there is a vertex u of P incident with edges that are not part of P. Consider the induced subgraph constructed by simply removing all edges that are part of P: $H \stackrel{\text{def}}{=} G - E(P)$. Note that H may be disconnected. Because every vertex in G has even degree, but also every vertex in P, so will every vertex in H have even degree. Let component H' contain u. Again, construct a (closed) trail P' in H' originating in u until no more edges can be added that are not yet contained in P'. Because $|E(P')| > 0$, merging P and P' will yield a larger trail in G. If this larger trail does not contain all edges of G, we repeat the procedure until we have constructed a closed trail containing all edges of G. This trail will form an Euler tour.

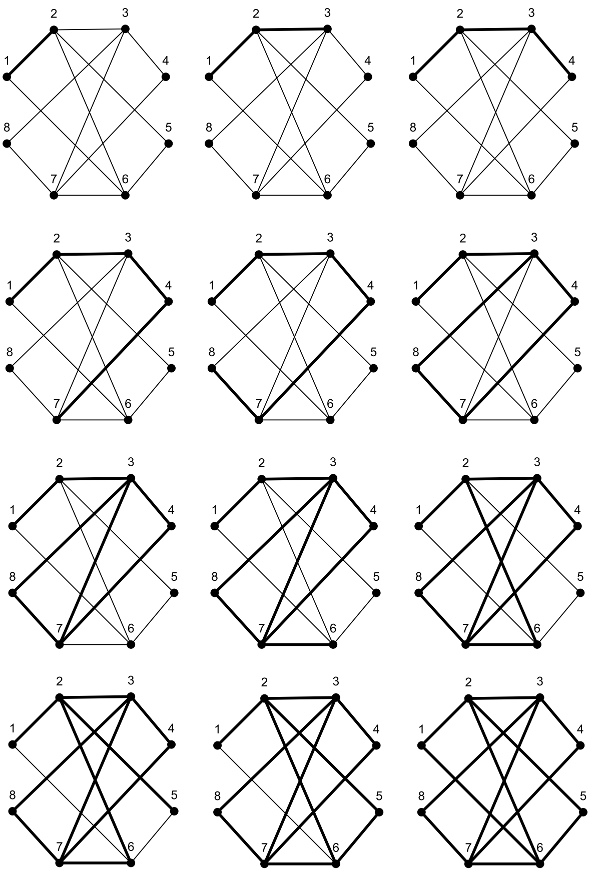
Q 97: Prove that a connected graph G (with more than one vertex) has an Euler trail if and only if it has exactly two vertices of odd degree. Moreover, the trail originates and ends in the vertices of odd degree.

First, let P be an Euler trail originating in u and ending in v. By the same reasoning as in the previous proof, all vertices except u and v must be of even degree.

Conversely, assume G has exactly two vertices u and v of odd degree. Consider the graph G^* constructed from G by adding an edge $e = \langle u, v \rangle$. All vertices in G^* will now have even degree. Because G^* is obviously also connected, we know that G^* has an Euler tour P. Removing e from P yields an Euler trail for G.

Q 98: Using Fleury's algorithm, obtain an Euler tour for the following graph:

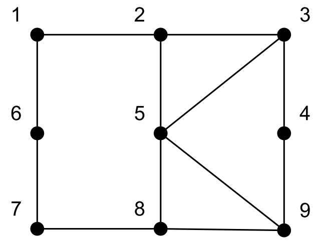




Q 99: If the number of odd-degree vertices in a connected graph $G = (V, E)$ is $2k$, show that the set E can be partitioned into k subsets such that the edges in each subset constitute a trail between two odd-degree vertices.

Suppose the odd-degree vertices are $v_i (1 \leq i \leq k)$ and $w_i (1 \leq i \leq k)$. Construct k new vertices $x_i (1 \leq i \leq k)$ and $2k$ new edges $\langle x_i, v_i \rangle$ and $\langle x_i, w_i \rangle$ for $1 \leq i \leq k$. In the graph G^* thus constructed, each vertex has even degree, so G^* is Eulerian. Construct an Eulerian tour Q in G^*. Observe that in this tour, the two edges adjacent to each new vertex x_i appear consecutively in Q for each i. Now delete from this circuit all the new vertices x_i (and, of course, all the new edges). The remaining edges in Q precisely constitute k pairwise disjoint sets, forming a partition of E such that the edges in each subset of the partition constitute a trail between two distinct *odd vertices.*

Q 100: Extend the following graph by adding a minimal number of edges such that the extended graph is simple and Eulerian.



We first need to identify the odd-degree vertices, and subsequently join pairs of vertices. In our case, adding $\langle 2,8\rangle$ *and* $\langle 3,9\rangle$ *does the trick.*

Q 101: Prove that a weakly connected digraph is Eulerian if and only if the indegree of each vertex is equal to its outdegree.

Whenever a walk passes through a vertex, two distinct arcs are used: one to the vertex and one from the vertex. So each such passing results in a contribution of one to the outdegree and one to the indegree. Thus the existence of a directed walk containing all the arcs implies that the outdegree of each vertex equals its indegree.

Conversely, let G be a weakly connected digraph with m arcs in which the indegree of each vertex equals its outdegree, We prove by induction on m that the digraph is Eulerian. The result is obviously true when $m = 2$. *Assume that the theorem is true for all weakly connected digraphs with* $m > 2$ *arcs, and let D be one such digraph. Observe that the outdegree (and therefore the indegree) of each vertex in D is positive. Let T be any trail from an arbitrary vertex u to another vertex v in this digraph. There will be at least one arc* $\langle \overrightarrow{v,w} \rangle$ *that is not an arc in T. So it is possibe to extend the trail and end up with a trail that terminates at u. If the closed trail* T^* *thus obtained contains all the arcs of D, we are done. Otherwise, delete from D all the arcs belonging to* T^* *as well as the vertices that become isolated as a result of this deletion. Each component of the resulting digraph* D^* *is weakly connected with fewer arcs in which the outdegree and the indegree are the same. So by the induction hypothesis, each component has a directed Eulerian circuit. Since D is weakly connected, each weak component of D has a vertex in common with the closed walk* T^*. *An Eulerian circuit for D can now be constructed by inserting an Eulerian circuit of each weak component H of* D^* *at a vertex common to both H and* T^*.

Q 102: If every vertex in a graph G has even degree, no edge in that graph is a cut edge.

If all vertices have even degree, G has an Euler tour, which, in turn, is the union of edge-disjoint cycles. In other words, every edge is part of a cycle. Therefore, no edge is a cut edge.

Q 103: Find all positive integers n such that K_n is Eulerian.

The degree of any vertex in a complete graph with n vertices is $(n-1)$*, so the graph is Eulerian if and only if n is odd.*

Q 104: Show that a digraph that has an Euler tour is a strongly connected digraph. Is the converse true?

Since the digraph is Eulerian, there is a closed directed trail emanating from every vertex that returns to it after traversing through each arc exactly once. Such a closed trail no passes through each vertex of the graph at least once. So there is a directed trail (and therefore a directed path) from every vertex to every other vertex, establishing the strong connectivity of the digraph.
The converse is not true; a strongly connected digraph need not be an Eulerian graph. As a counterexample, consider the digraph G obtained by introducing a new arc from a vertex to a nonadjacent vertex in the cyclic digraph $1 \to 2 \to 3 \to 4 \to 1$.

Q 105: Consider a graph G with n vertices and m edges.

(a) Can G be Eulerian if n is even and m is odd?

(b) Can G be Eulerian if n is odd and m is even?

(a) Let C is a cycle with an even number of vertices in which v is a vertex. Consider a cycle C^* *with an odd number of vertices passing through v such that the two cycles have no edge in common. The tour G that consists of the edges of these two cycles is a graph in which each vertex is even.*

(b) In part (a), suppose both C and C^* *are odd-length cycles. The tour constructed as the union of the two cycles has an odd number of vertices and even number of edges.*

Q 106: If graph G is Hamiltonian, then for every proper nonempty subset $S \subset V(G)$, we have that $\omega(G-S) \leq |S|$.

Consider a Hamilton cycle C of G. If we consider any proper nonempty subset $S \subset V(G)$*, then obviously, because every vertex is visited exactly once, the number of components in* $C - S$ *will be less or equal to* $|S|$*. However, because C contains* *all vertices of G, we also have that $\omega(G - S) \leq \omega(C - S)$, which completes the proof.*

Q 107: Prove that if G is a simple graph with $n = |V(G)|$ vertices, $n \geq 3$ and each vertex v has degree $\delta(v) \geq n/2$, then G is Hamiltonian.

Assume the theorem is false. Let G be a non-Hamiltonian graph with $n \geq 3$ vertices and for which $\delta(v) \geq n/2$ for each of its vertices. Moreover, assume that G has a maximal number of edges, i.e., adding a single edge (while keeping G simple) would make it Hamiltonian. Let u and w be two nonadjacent vertices. By construction of G we know that if we add an edge $e = \langle u, w\rangle$, the resulting graph G^* would be Hamiltonian, and thus there exists a Hamilton path (u, w)-path P in G with $u = [v_1, v_2, \ldots, v_n = w]$.
Now consider the following two sets of vertices:

$$S = N(u) = \{v_i | \langle u, v_i\rangle \in E(G)\} \textit{ and } T = \{v_i | \langle v_{i-1}, w\rangle \in E(G)\}$$

S consists of the neighbors of u, whereas T consists of the successors on P of neighbors of w. Note that $|S| \geq n/2$. Likewise, because P contains all vertices in G, T contains as many elements as there are edges $\langle v_{i-1}, w\rangle$, which corresponds to $\delta(w)$. This means that $|T| \geq n/2$. Furthermore, vertex u is not contained in S (because it cannot be a neighbor of itself), nor is it contained in T (which contains only successors of other vertices on P). In other words, $S, T \subseteq \{v_2, \ldots, v_n\}$, which, together with the fact that $|S| + |T| \geq n$, means that the two sets have at least one vertex in common. Let this be vertex v_j. We now have the situation that v_j is a neighbor of u, and that v_j's predecessor v_{j-1} is a neighbor of w. But in that case, we can construct the Hamilton cycle $[u = v_1, v_j, v_{j+1} \ldots v_n = w, v_{j-1}, v_{j-2} \ldots v_1 = u]$. Note that this cycle does not contain edge $\langle u, w\rangle$. In other words, we have just shown that G is Hamiltonian, which contradicts our initial assumption. This means that there is no vertex $v_j \in S \cap T$ and thus $|S \cap T| = 0$. On the other hand, we know that $u \notin S \cup T$, so that $|S \cup T| < n$. This now brings us to:

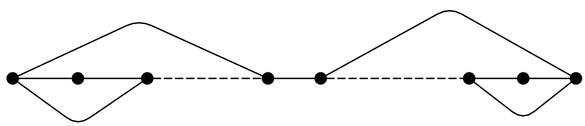
$$\delta(u) + \delta(w) = |S| + |T| = |S \cup T| + |S \cap T| < n$$

which cannot be true, meaning that we cannot assume the theorem is false.

Q 108: Let G be a non-Hamiltonian, connected graph. For every pair of nonadjacent vertices u and v, $\delta(u) + \delta(v) \geq k$, for some $k > 0$. Show that G contains a path of length k.

Let $P = [v_0, v_1, v_2, \ldots, v_p]$ be a longest path in the graph. Since P is a longest path, neither v_0 nor v_p can be adjacent to a vertex not in P. Let v_0 be adjacent to intermediate vertex v_i. We claim that vertex v_p cannot be adjacent to vertex v_{i-1}. For suppose that this were the case. Then the p vertices in the path constitute the cycle $C = [v_0, v_i, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1, v_0]$, which cannot contain all

vertices of G, beacause G is not Hamiltonian. This implies that there is a vertex w that is not a vertex in the cycle but that is adjacent to one of its vertices. This, in turn, implies that there is a path of length $(p+l)$ in the graph violating the maximality of P. Since v_p is adjacent to v_{p-1}, we conclude that v_0 cannot be adjacent to v_p.
Note again that because P is a longest (v_0, v_p)-path, that each neighbor of v_0 must lie on P, for otherwise we could have constructed a longer path. The same holds for v_p. Also, we have shown that if v_0 is adjacent to v_i, then v_p cannot be adjacent to v_{i-1}, leading to the following general organization:



This means that $\delta(v_0) + \delta(v_p) \leq p$. But $\delta(v_p) + \delta(v_0) \geq k$ since the two terminal vertices of the longest path are nonadjacent. Since there is a path with p edges and since $p \geq k$, there exists a path with k edges.

Q 109: If G is a connected graph with k odd-degree vertices, find the minimum number of trails in G such that every edge in the graph is an edge in exactly one of these trails.

For each odd-degree vertex v, we know that any trail that passes through v will not contain all of its incident edges. Hence, to make sure that we have those edges in a trail as well, we need to make sure that every trail starts and ends in a different odd-degree vertex. Therefore, we need at least $k/2$ trails. (Note, by the way, that k is even.)

Q 110: Suppose in a group of n people $(n > 3)$, any two of them together know all the other people in the group. Show that these n people can be seated around a circular table so that each person is seated between two acquaintances.

For this problem, we need to use the following property: if for any two vertices u and v in a non-Hamiltonian connected graph, $\delta(u) + \delta(v) \geq k$, the graph contains a path of length k. In our example, consider the acquaintance graph. We know that $\delta(u) + \delta(v) \geq n$, meaning that there is a path of length n, which in our case coincides with a cycle of length n. This is the cycle we're looking for.

Q 111: Show that a directed graph D is Hamiltonian if and only if its transformed undirected version $\hat{D}$ is Hamiltonian.

First assume that D is Hamiltonian and let $C = [v^1, v^2, \ldots, v^n, v^1]$ be a Hamilton *cycle. Clearly, the cycle*

$$\hat{C} = [v^1, v^1_{out}, v^2_{in}, v^2, v^2_{out}, \ldots, v^n_{in}, v^n, v^n_{out}, v^1_{in}, v^1]$$

is a Hamilton cycle in $\hat{D}$.
Conversely, consider a Hamilton cycle $\hat{C}$ in $\hat{D}$. Obviously, for each vertex $v^k \in V(\hat{D})$, $\hat{C}$ contains the edges $\langle v^k_{in}, v^k \rangle$ and $\langle v^k, v^k_{out} \rangle$, for otherwise it would be impossible to have visited vertex v^k. For this reason, $\hat{C}$ corresponds to a unique directed Hamilton cycle C in D.

Q 112: Show that a k-regular simple graph with $2k-1$ vertices is Hamiltonian.

Dirac's theorem tells us that for a simple graph G with $n \geq 3$ vertices, G is Hamiltonian if $\delta(v) \geq n/2$ for each vertex v. In the case of a k-regular graph with $2k-1$ vertices, we have that $k \geq (2k-1)/2 = k - \frac{1}{2}$, meaning that it is Hamiltonian as well.

Problems Chapter 5

Q 113: Prove that for any spanning tree T of a graph G and edge $e = \langle u, v \rangle \in E(G)$ that is not in T, $T + e$ contains a unique cycle.

Note that because T is acyclic, connected, and spanning, we necessarily have that every cycle in $T + e$ contains e. We also know that C is a cycle of $T + e$ if and only if $C - e$ is a path connecting vertices u and v. But as we have proven before, $C - e$ is the unique path connecting u and v in T, and thus C must be unique.

Q 114: In Kruskal's algorithm, we select an edge $\hat{e}$ of the cycle C such that $\hat{e} \notin E(T_{opt})$, but $\hat{e} \in E(T)$. Prove that $\hat{e}$ indeed exists.

Prove this by contradiction, i.e., assume that such an edge does not exist. In that case, all edges of C are also edges of T_{opt}, and none is edge in T. If this were true, then T_{opt} would contain the cycle C, which is clearly impossible.

Q 115: Describe Dijkstra's algorithm for constructing a sink tree using pseudo-code, analogously to the description found in Chapter 3.

$S(u) \leftarrow \{u\}$
$\mathbf{L}(u) \leftarrow (u, 0)$; *for each* $v \in V(G), u \neq v : \mathbf{L}(v) \leftarrow (-, \infty)$;
while $S(u) \neq V$ ***do***
  $R(u) \leftarrow S(u) \cup_{v \in S(u)} N_{in}(v)$;
  for all $y \in R(u) \backslash S(u)$ ***do***
    for all $x \in N_{out}(y) \cap S(u)$ ***do***
      if $L_2(y) + w(\langle \overrightarrow{y, x} \rangle) < L_2(y)$ ***then***
        $\mathbf{L}(y) \leftarrow \big(x, L_2(x) + w(\langle \overrightarrow{y, x} \rangle)\big)$
      end if
    end for
  end for
  select $v \notin S(u)$ ***where*** $L_2(v)$ *is minimal;*
  $S(u) \leftarrow S(u) \cup \{v\}$;
end while

Q 116: Prove that for any connected graph G with n vertices and m edges, $n \leq m + 1$.

The proof proceeds by induction on the number of edges m. Clearly, if $m = 1$, we necessarily have $n = 2$ so that the theorem is true. Now assume the theorem is true for all graphs with fewer than k edges and consider a graph G with exactly k edges and n vertices.
Suppose that G contains a cycle C. In that case, choose an arbitrary edge $e \in E(C)$ and construct the induced subgraph $G^* = G - e$. Because e was lying on the cycle C, G^* will still be connected, meaning that $n = |V(G^*)| \leq |E(G^*)| + 1 =$

$(k-1)+1=k$. But in that case, we certainly have that $n \leq k+1$.
If G does not contain a cycle, find a longest path P in G. Let u and w be the end points of P. Note that the degree of each these nodes must be 1, for otherwise P could not have been a longest path. Now consider the induced subgraph $G^* = G - u$. Clearly, G^* is connected and we have $|V(G^*)| = n-1$ and $E(G^*) = k-1$. By induction, we thus also have that $n-1 \leq (k-1)+1 = k$, and thus $n \leq k+1$, completing our proof.

Q 117: Show by using a proof by induction that a tree with n vertices has exactly $n-1$ edges.

If $n=1$, the graph cannot have any edges or there would be a loop, with the vertex connecting to itself, so there must be $n-1=0$ edges.
Suppose that every tree with k vertices has precisely $k-1$ edges. If the tree T contains $k+1$ vertices, we will show that it contains a vertex with a single edge connected to it. If not, start at any vertex, and start following edges marking each vertex as we pass it. If we ever come to a marked vertex, there is a loop in the edges which is impossible. But since each vertex is assumed to have more than one vertex coming out, there is never a reason that we have to stop at one, so we much eventually encounter a marked vertex, which is a contradiction.
Take the vertex with a single edge connecting to it, and delete it and its edge from the tree T. The new graph T' will have k vertices. It must be connected, since the only thing we lopped off was a vertex that was not connected to anything else, and all other vertices must be connected. If there were no loops before, removing an edge certainly cannot produce a loop, so T' is a tree. By the induction hypothesis, T' has $k-1$ edges. But to convert T' to T we need to add one edge and one vertex, so T also satisfies the formula.

Q 118: Prove that a connected graph G with n vertices and m edges for which $n = m+1$, is a tree.

We prove the theorem by contradiction. To this end, assume G is not a tree, i.e., it contains a cycle C. Let edge $e \in E(C)$. Obviously, the induced subgraph $G-e$ is still connected, but with one edge less than G. We know that $|V(G-e)| \leq |E(G-e)|+1$. With $|V(G-e)| = n$ and $|E(G-e)| = m-1$, we thus have that $n \leq (m-1)+1 = m$. However, we assumed that $n = m+1$, which contradicts that $n \leq m$. Hence, our initial assumption, namely that G is not a tree, was false.

Q 119: Show that a graph G is a tree if and only if there exists exactly one path between every two vertices u and v.

We need to prove two things: (1) If G is a tree then there exists a unique path between every two vertices and (2) if there exists a unique path between every two vertices, then G is tree.

(1) *Let G be a tree and let u and v be two distinct vertices. Because G is connected, there exists a (u, v)-path P. Assume there is another, distinct (u, v)-path Q. Let x be the last vertex on P that is also on Q when traversing P starting from u. In other words, the next vertex following x will be different for P and for Q. Likewise, let y be the first vertex succeeding x that is common to both P and Q again. We have now identified a cycle in G, contradicting that G was a tree.*

(2) *Now assume that G is not a tree. Note that because there is a path between every two vertices, G is connected. If G is not a tree, there must be a cycle $C = [v_1, v_2, \ldots, v_n = v_1]$. Clearly, for every two distinct vertices v_i and v_j ($i < j$) on C we have also have two distinct (v_i, v_j)-paths: $P_1 = [v_i, v_{i+1}, \ldots, v_{j-1}, v_j]$ and $P_2 = [v_i, v_{i-1}, \ldots, v_2, v_1 = v_n, v_{n-1}, \ldots, v_{j+1}, v_j]$, which contradicts the uniqueness of paths.*

Q 120: Prove that an edge e of a graph G is a cut edge if and only if e is not part of any cycle of G.

Again, we need to prove two things: (1) If e is not part of any cycle, then e is a cut edge, and (2) if e is a cut edge, it cannot be part of any cycle of G.

(1) *By contradiction: assume that $e = \langle u, v \rangle$ is not a cut edge (and not part of any cycle). If e is not a cut edge, then u and v must still be in the same component of $G - e$. This implies that there is a (u, v)-path P in $G - e$ connecting u and v. However, this also means that $P + e$ is a cycle in G, which violates our assumption.*

(2) *Again, by contradiction: let $e = \langle u, v \rangle$ be a cut edge of G and let x and y be two vertices in different components of $G - e$. Because there is an (x, y)-path P in G connecting x and y, we necessarily have that e is part of P. Assume that u precedes v when traversing P from x to y. Let P_1 be the (x, u)-path part of P and P_2 the (v, y)-path that is part of P. If e were part of a cycle C, then u and v would be connected in $G - e$ through the path $C - e$. Let u^* be the first vertex common to P_1 and $C - e$ when traversing P_1 from x. Likewise, let v^* be the first vertex common to P_2 and $C - e$ when traversing P_2 from y. Let $a \xrightarrow{Q} b$ denote that part of path Q that connects vertex a to b. Clearly, the path $x \xrightarrow{P_1} u^* \xrightarrow{C-e} v^* \xrightarrow{P_2} y$ connects x and y in $G - e$, contradicting that e was a cut edge. Hence, e cannot be part of any cycle.*

Q 121: Prove that a connected graph G is a tree if and only if every edge is a cut edge.

Again we need to prove two things: (1) If G is a tree then every edge is a cut edge, and (2) if every edge is a cut edge, then G is a tree.

(1) *Let G be a tree and e an edge of G. Because G contains no cycles, e is also not contained in any cycle, meaning that it must be a cut edge.*

(2) *Assume G contains a cycle C. However, we now know that none of the edges of C can be a cut edge, which means that not every edge in G is a cut edge, contradicting our starting-point.*

Q 122: Show that any tree with at least two vertices is bipartite.

A tree has no cycles, so it certainly does not contain any cycles of odd length. Therefore, it is bipartite.

Q 123: Show that a graph G is a tree if and only if it is acyclic and whenever any two vertices u and v in G are joined by an edge, the graph $G^* = G + \langle u, v \rangle$ has exactly one cycle.

If G is a tree, it is connected and acyclic. Let u and v be any two nonadjacent vertices in G. There is a unique path between u and v. If we join u and v by an edge, this edge and path P create a unique cycle in the enlarged graph G^*. *On the other hand, suppose G is an acyclic graph in which u and v are two any arbitrary nonadjacent vertices such that the linking of the two by a new edge creates a unique cycle in* G^*. *This implies that there is a path in G between u and v. So G is connected and hence is a tree.*

Q 124: Prove that a graph is connected if and only if it has a spanning tree.

Let G be a connected graph. Delete edges from G that are not cut edges until we get a connected subgraph H in which each edge is a cut edge. Then H is a spanning tree. On the other hand, if there is a spanning tree in G, there is a path between every pair of vertices in G: thus G is connected.

Q 125: Show that if a graph is disconnected, its complement is connected.

If a graph G is not connected, it will have at least two components. Suppose u and v are two vertices belonging to two different components of G. Then these two vertices are adjacent in the complement of the graph. In other words, G and its complement cannot both be disconnected graphs. So whenever G is a disconnected graph, its complement is necessarily a connected graph.

Q 126: Show that every tree of $n \geq 2$ vertices has at least two vertices having degree 1.

Suppose the degrees of the n vertices of a tree are d_i, *where* $i = l, 2, ..., n$. *Then* $d_1 + d_2 + ... + d_n = 2n - 2$. *If each degree is more than 1, the sum of the n degrees is at least* $2n$. *So there is at least one vertex (say vertex 1) with degree 1. Then* $d_2 + d_3 + ... + d_n = 2n - l$. *At least one of these* $(n - 1)$ *positive numbers is* *necessarily 1. So there is one more vertex of degree 1. Thus at least two of the degrees must be 1.*

Q 127: Show that the sequence $\mathbf{d} = [d_1, d_2, \ldots, d_n]$ of positive integers, where $d_1 \leq d_2 \leq \ldots \leq d_n$ is the degree sequence of a tree with n vertices if and only if $\sum_{i=1}^{n} d_i = 2(n-1)$.

The necessity is obvious. We prove the sufficiency by induction on n. The property holds for $n = 1$ *and* $n = 2$. *Assume that the property holds* $n \geq 3$. *Let* $0 < d_1 \leq d_2 \leq \ldots \leq d_n$ *and* $d_1 + d_2 + \cdots + d_n = 2(n - l)$. *At least one of these numbers is 1. So* $d = 1$. *Also* $d_n > 1$. *Let* $d^* = d_n - 1$. *Then* $d_2 + \cdots + d_{n-1} + d^* = 2(n-2)$. *So by the induction hypothesis, there exists a tree T with* $(n - l)$ *vertices and degrees* $d_2, d_3, \ldots, d_{n-1}$ *and* d^*. *Construct a new vertex x and join that to the vertex of degree* d'. *Now we have a tree with n vertices with degrees* $1, d_2, \ldots, d_n$. *Thus the property hold for n.*

Q 128: Show that the number of vertices in a binary tree is odd.

Every vertex other than the root is an odd vertex. The number of odd vertices is even. If we now include the root also, the total number of vertices is odd.

Q 129: Show that the number of terminal vertices in a binary tree with n vertices is $(n + 1)/2$.

Suppose there are k terminal vertices. Then the sum of the degrees of the n vertices is $k + 2 + 3(n - k - 1)$, *which is equal to* $2(n - 1)$ *since the graph is a tree. Thus* $k = (n + 1)/2$.

Q 130: Let $\delta_{min}(G)$ denote the minimal vertex degree of graph G. Furthermore, let C_n denote the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{\langle v_1, v_2\rangle, \langle v_2, v_3\rangle, \ldots, \langle v_n, v_1\rangle\}$, i.e., a cycle of length n.

(a) Show that if T is a tree with n vertices and G is a graph with $\delta_{min}(G) \geq (n - 1)$, T is isomorphic to a subgraph of G.

(b) Show that a tree with n vertices is isomorphic to a subgraph of the complement of C_{n+2}.

(a) The proof is by induction on n. This is true when the tree has two vertices. The induction hypothesis is that if T^* *is any tree with* $(n - 1)$ *vertices and* G^* *is any graph with* $\delta_{min}(G^*) \geq (n - 2)$, *then* T^* *is isomorphic to a subgraph of* G^*. *Let T be any tree with n vertices, and let G be any graph with* $\delta_{min}(G) \geq (n - 1)$. *Let v be any terminal vertex in T, and let u be the vertex adjacent to v in T. Then* $T - v$ *is a tree with* $(n - 1)$ *vertices. Moreover,* $\delta_{min}(G) \geq (n - 1) > (n - 2)$. *So by the induction hypothesis, the tree* $T - v$ *is isomorphic to a subgraph of G. Let* u^* *be the vertex in G that corresponds (for*

this isomorphism) to vertex u. Then $\delta(u^*) \geq (n-1)$ in G. The graph $T - v$ has only $n-2$ vertices in addition to vertex u. So there should be a vertex w in G that is adjacent to u^* such that w does not correspond to any vertex in $T - v$. By identifying v with vertex w, we see that T is isomorphic to a subgraph of G. Thus the theorem is true for n as well.

(b) *The complement of C_{n+2} is an r-regular graph G, where $r = n > n - l$. In other words, $\delta_{min}(C_{n+2}) \geq n-1$, so that a tree with n vertices is isomorphic to a subgraph of C_{n+2}.*

Q 131: Show that if $T_i = (V_i, E_i)$, where $i = 1, 2, ..., k$ are subtrees of $T = (V, E)$ such that every pair of subtrees have at least one vertex in common, the entire set of subtrees have a vertex in common.

Let n be the number of vertices of T. The proof is by induction on n. The desired property holds if $n = 2$. Assume that the property holds for all trees with $n > 2$ vertices.
Let T be a tree with $(n+1)$ vertices in which x is a leaf vertex adjacent to a vertex y. Suppose the subtrees $T_1, T_2, \ldots, T_k$ of T are such that every pair of them has at least one vertex in common. If x is not a vertex in any of these trees, the trees are subtrees of a tree with n vertices; thus the property holds for the graph T with $(n+1)$ vertices. If one of these trees is the tree with just one vertex x, x is common to all the trees; thus the property holds in this case.
We now examine the remaining case. Consider $T_i - x$. If x is a vertex common to T_i and T_j, y is also a vertex common to T_i and T_j. Therefore, y is a common vertex for $T_i - x$ and $T_j - x$. Thus by the induction hypothesis, all the subtrees $T_i - x$ have a common vertex. Therefore, the entire collection $\{T_i\}$ has a vertex in common.

Q 132: If both G and its complement are trees, how many edges does G have?

The total number of edges of G and $\overline{G}$ is equal to $n(n-1)/2$. We also know that $|E(G)| = |E(\overline{(G)})| = n-1$. Therefore, $(n-1)+(n-1) = n(n-1)/2$, which gives is $n = 4$.

Q 133: A forest is a graph consisting of k components, each component being a tree. How many edges does a forest of n vertices and k trees have?

Let each component T_i have n_i vertices and $n_i - 1$ edges. We then know that $|E(G)| = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1 = n - k$.

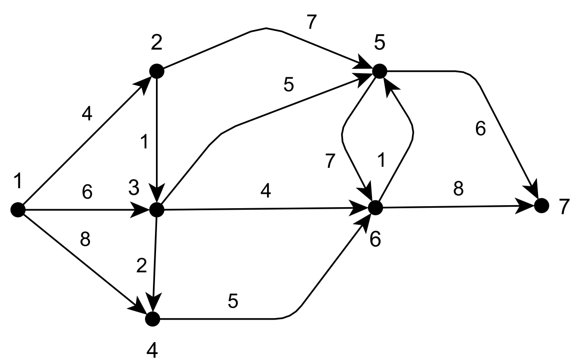
Q 134: Show that if the degree of every non-leaf vertex in a tree is 3, the number of vertices in the tree is even.

Let k be the number of leaf vertices. We then know that $k + 3(n-k) = 2(n-1)$, which means that $n = 2k - 2$ and thus even.

Q 135: If the degree of each vertex in a graph is at least two, show that there is a cycle in the graph.

If there is a loop at a vertex, that loop can be considered a cycle. If there is more than one edge between two vertices, any two edges joining two vertices will form a cycle. Suppose the graph is simple. Let v_0 be any vertex in the graph, and let e_1 be the edge joining this vertex and vertex v_1. Now there exists a third vertex v_2 and edge e_2 joining v_1 and v_2. This process of finding new vertices and edges is repeated, and at the k^{th} stage, we have edge e_k joining vertices v_{k-1} and v_k and a path from v_0 to v_k consisting of k edges. Since the number of vertices in the graph is finite, we must ultimately choose a vertex that has been chosen before. Suppose v, is the first repeated vertex in this process. Then the path between the two occurrences of this repeated vertex is a cycle.

Q 136: Using Dijkstra's algorithm, find the sink tree rooted at vertex 7.



Using the notation $v(u,x)$ to denote that vertex v can reach vertex 7 via u at a distance of x, we obtain:

1. $S_0(7) = 7$
2. $S_1(7) = \{5(7,6), 7(7,0)\}$
3. $S_2(7) = \{5(7,6), 6(5,6), 7(7,0)\}$
4. $S_3(7) = \{3(5,11), 5(7,6), 6(5,6), 7(7,0)\}$
5. $S_4(7) = \{3(5,11), 4(6,11), 5(7,6), 6(5,6), 7(7,0)\}$
6. $S_5(7) = \{2(3,12), 3(5,11), 4(6,11), 5(7,6), 6(5,6), 7(7,0)\}$
7. $S_6(7) = \{1(2,16), 2(3,12), 3(5,11), 4(6,11), 5(7,6), 6(5,6), 7(7,0)\}$

Q 137: List the edges of a sink tree rooted at vertex 1 of the network with $V = \{1,2,3,4,5,6\}$ and $E = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,5\rangle, \langle 3,6\rangle, \langle 4,5\rangle, \langle 5,6\rangle\}$ with weights 4, 7, 3, 3, 2, 2, 3, and 2, respectively.

$\{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 4,5\rangle, \langle 5,6\rangle\}$.

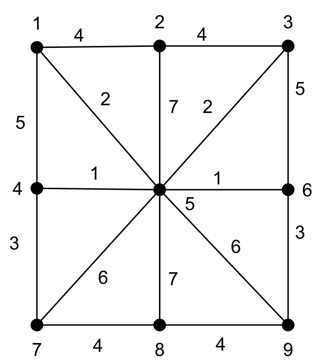
Q 138: If no two edge weights of a connected graph G are equal, show that G has a unique minimum spanning tree.

Suppose there are two minimum spanning trees, T and T^* with $w(T) = w(T^*) = s$. Let $e = \langle u,v\rangle \in E(T)$. We know that $T - e$ divides T into two subgraphs, and thus partitions the set of vertices into two sets V_1 and V_2. Because T^* is a spanning tree, there is (u,v)-path in T^* that will have an edge $e^* = \langle p,q\rangle$ with $p \in V_1$ and $q \in V_2$. Now consider the trees $T_1 = T - e + e^*$ and $T_2 = T^* - e^* + e$. We know that $w(T_1) = s - w(e) + w(e^*)$, and $w(T_2) = s - w(e^*) + w(e)$. If $w(e) > w(e^*)$, $w(T_1) < s$, which is a contradiction since T is a minimal spanning tree. If $w(e) < w(e^*)$, $w(T_2) < s$, which is also a contradiction. We conclude that there is only one minimal spanning tree.

Q 139: Show that if a connected weighted graph G contains a unique edge e of minimum weight, e is an edge of every minimal spanning tree of G.

Suppose T is a minimal spanning tree of G and e is not an edge of T. Let f be any edge of a cycle of minimal weight also containing e. Then $T^* = T - f + e$ is spanning tree of G, and $w(T^*) = w(T) - w(f) + w(e)$. Since $w(e) < w(f)$, $w(T^*) < w(T)$, and this contradicts the assumption that T is a minimal spanning tree.

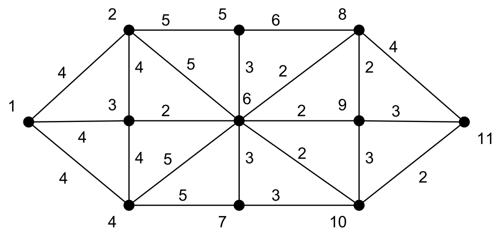
Q 140: Find the weight of a minimum spanning tree in the following graph, using Kruskal's algorithm.



Add, in order, the edges $\langle 4,5\rangle, \langle 5,6\rangle, \langle 1,5\rangle, \langle 3,5\rangle, \langle 4,7\rangle, \langle 6,9\rangle, \langle 1,2\rangle, \langle 7,8\rangle$, lead-

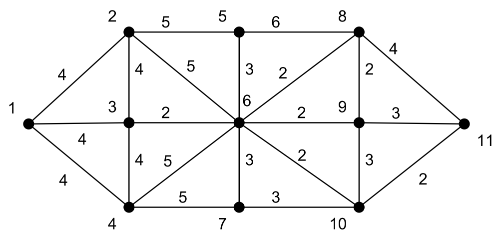
ing to a total weight of 20.

Q 141: Obtain a minimum spanning tree for the following graph, using Kruskal's algorithm.

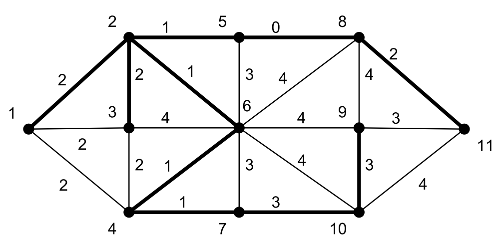


Add in sequence the following edges: $\langle 6,8\rangle, \langle 6,9\rangle, \langle 6,10\rangle, \langle 3,6\rangle, \langle 10,11\rangle, \langle 5,6\rangle, \langle 6,7\rangle, \langle 2,3\rangle, \langle$

Q 142: Construct a *maximum* weight spanning tree for the following graph:



In this case, simply use Kruskal's algorithm on the same graph with each weight $w(e)$ *replaced by* $\max\{w(e)|e \in E(G)\} - w(e)$. *This gives a minimal spanning tree T which corresponds to a maximal-weight spanning tree with weight* $\max\{w(e)|e \in E(G)\} - w(T)$.



Problems Chapter 6

Q 143: Given a connected, simple undirected graph G with n vertices. Argue that a given vertex u can lie on at most $(n-1)(n-2)/2$ paths connecting other (distinct) vertices.

There are a total of $\binom{n-1}{2}$ different pairs of vertices, other than u. Because the graph is connected, all pairs of vertices are connected through a path. Hence, At most $\binom{n-1}{2}$ paths will also pass through u.

Q 144: Prove that the center of a tree is either a singleton set consisting of a unique vertex or a set consisting of two adjacent vertices.

If a tree has two vertices, the center is the set of those two vertices. If there are three vertices in a tree, the center is the set consisting of the nonterminal vertex (i.e., the vertex u with $\delta(u) > 1$). A tree with four vertices is either $K_{1,3}$ (with three terminal vertices) or a path with two terminal vertices. ln the former case, the cardinality of the center is 1; in the latter case, the center is the set of two adjacent nonterminal vertices. More generally, let T be a tree with five or more vertices, and let T^* be the tree obtained from T by deleting all terminal vertices of T simultaneously. Observe that the eccentricity of any vertex in T^* is one less than the eccentricity of that vertex in T. Thus the center of T is equal to the center of T^*. If the process of deleting terminal vertices is carried out successively, we finally have a tree with four or fewer vertices.

Q 145: A path P between two distinct vertices in a connected graph G is a diametral path if there is no other path in G whose length is more than the length of P. Show that (a) every diametral path in a tree will pass through its central vertices, and (b) the center of a tree can be located once a diametral path in the tree is discerned.

Let t be the length of any diametral path in a tree, and let P be a fixed diametral path joining the vertices v and w.

(a) If t is even, there exists a unique vertex c in P that is equidistant from either v or w. In this case, c is a central vertex. Suppose Q is another diametral path. Since the graph is connected, the two diametral paths should have a vertex in common. If c is not a common vertex, it is possible to obtain path whose length is more than t. So if the length of a diametral path is even, there exists a unique central vertex on that path through which every diametral path passes.

(b) If t is odd, there exist two vertices c' and c'' in P such that the number of edges in the path between v and c' is equal to the number of edges between w and c''. In this case, both c' and c'' are central vertices. Suppose Q is another diametral path. Then both P and Q share the edge joining c' and c'' as a common edge.

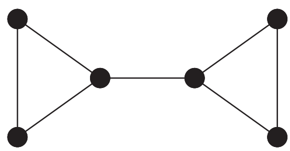
Thus once a diametral path in a tree is located, it is easy to find the center of the tree.

Q 146: Show that the weighted clustering coefficient is identical to th clustering coefficient in an unweighted graph for the special case that all weights are equal to 1.

First, note that the vertex strength, defined as $\sum_{e=\langle v,w\rangle} w(e)$ over all edges incident with v is exactly the same as $\delta(v)$ in case all weights are 1. Taking a closer look at the nominator of the weighted clustering coefficient, we see that we are enumerating over $\binom{\delta(v)}{2}$ pairs of edges, and each time for a total weight of 2. Furthermore, note that we are considering only those edges $e_{i,j}$ and $e_{i,k}$ for which we know that $\mathbf{A}[j,k] = 1$. Of course, there are exactly $|E(G[N(v)])|$ such edges, i.e., the number of edges in the graph induced by the neighbor set $N(v)$ of v. However, we are taking all pairs $e_{i,j}$ and $e_{i,k}$ twice into account. Hence, the nominator is equal to $4 \cdot |E(G[N(v)])|$. Finally, realizing that $\delta(v) = |N(v)|$, we have shown that the two clustering coefficients are the same in the case all weights are equal to 1.

Q 147: Give an example of a simple, undirected graph G for which $CC(G) \neq \rho(G)$. Consider the case that all vertices of G have at least degree 2.

For the following graph, we have that $CC(G) = \frac{7}{9}$, whereas its network density is equal to $\frac{7}{15}$



Q 148: Given an $ER(n,p)$ random graph. How many vertices can we expect to have vertex degree k?

We have already explained that the probability that a vertex has degree k is equal to $p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$. This means that we can expect a total of $n \cdot p_k$ vertices to have degree k.

Q 149: Prove that a triple is always connected.

We know that a simple graph with n vertices and $m+1$ edges is a tree, and thus connected. A triple is a tree, being a subgraph with 3 vertices and 2 edges, is a tree and is thus connected.

Q 150: Explain why the giant cluster of a $ER(2000, 0.015)$ shrinks after removing more than 98% of the vertices.

Removing 98% of the vertices means that there are only 40 vertices left. This graph *is so small that one cannot expect to see behavior that is similar to larger random graphs.*

Q 151: Show that $\sum_{i=1}^{m} i = \frac{1}{2}m(m+1)$, where we assume $m \geq 1$.

By induction. The equation is seen to hold for $m = 1$. Assume it to hold for values $m < M$ and consider the case $m = M + 1$. We have:

$$\sum_{i=1}^{M+1} i = \sum_{i=1}^{M} i + (M+1) = \frac{1}{2}M(M+1) + (M+1) = \frac{1}{2}(M+1)(M+2)$$

which completes the proof.

Q 152: Prove that $\sum_{k=m}^{n} \frac{1}{(k+1)(k+2)} = \frac{n-m+1}{(m+1)(2+n)}$.

By induction. Cleary, for $n = m$ the equation holds. Now assume it is true for $k < n$. In that case, we have:

$$\begin{aligned}
\textstyle\sum_{k=m}^{n} \frac{1}{(k+1)(k+2)} &= \left(\textstyle\sum_{k=m}^{n-1} \frac{1}{(k+1)(k+2)}\right) + \frac{1}{(n+1)(n+2)} \\
&= \frac{(n-1)-m+1}{(m+1)((n-1)+2)} + \frac{1}{(n+1)(n+2)} \\
&= \frac{1}{n+1}\left(\frac{n-m}{m+1} + \frac{1}{n+2}\right) \\
&= \frac{1}{n+1}\left(\frac{n^2-nm+2n-m+1}{(m+1)(n+2)}\right) \\
&= \frac{1}{n+1}\left(\frac{(n+1)(n-m+1)}{(m+1)(n+2)}\right) \\
&= \frac{(n-m+1)}{(m+1)(n+2)}
\end{aligned}$$