Graph Theory and Complex Networks: An Introduction

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Chapter 04: Network traversal

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Introduction

Algorithms that allow one to move or route through a network

1. Euler tours: visit every edge exactly once.
2. Hamilton cycles: visit every vertex exactly once.

Network traversal

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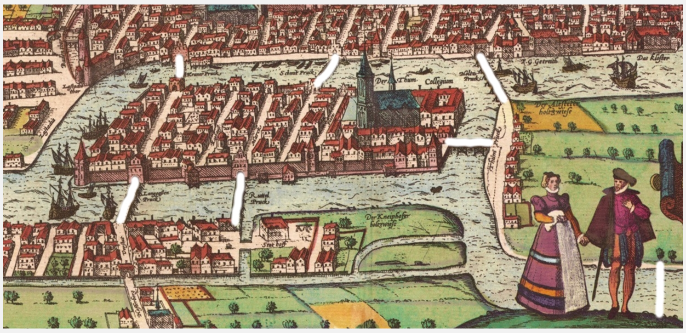
The Königsberg problem



Question

Can one walk through the city and cross each of the seven bridges exactly once?

Network traversal 4.1. Euler tours

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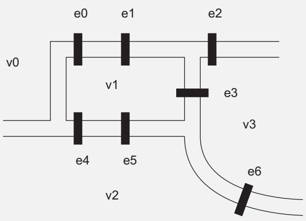
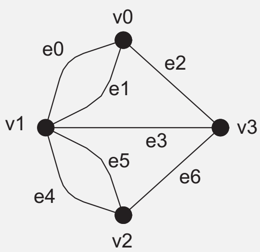
Modeling the problem in terms of graphs





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Euler tours

Definition

A tour of a graph G is a (u, v)-walk in which $u = v$ (i.e., it is a closed walk) and that traverses each edge in G. An Euler tour is a tour in which all edges are traversed exactly once.

Related: Chinese postman problem

- So called because originally formulated by a Chinese mathematician.
- **Issue:** Schedule the round of a postman such that (1) all streets are passed at least once and (2) the total traveled distance is minimal.
- **Solution:** Extend map of streets to a Eulerian graph with minimal weight.

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Necessary and sufficient conditions

Theorem

A connected graph G (with more than one vertex) has an Euler tour iff it has no vertices of odd degree.

Proof: Euler tour $\Rightarrow$ no odd-degree vertices

- Let C be an Euler tour starting/ending in vertex v. Let $u \neq v$
- $u \in V(C), \forall \langle w_{in}, u \rangle \in E(C) : \exists \langle u, w_{out} \rangle \in E(C)$.
- Every edge is traversed exactly once $\Rightarrow$ unique pairing of edges $\langle w_{in}, u \rangle$ and $\langle u, w_{out} \rangle$
- $\delta(u)$ must be even.

Necessary and sufficient conditions

Proof: no odd-degree vertices $\Rightarrow$ exists Euler tour

- Select v and construct trail P until you need to cross an edge for the second time. Let P end in w.
- Assume $w \neq v \Rightarrow$ entered w once more than left it $\Rightarrow \delta(w)$ is odd. Contradiction. Hence P must end in v.
- $E(P) = E(G) \Rightarrow$ done. Assume $E(P) \subset E(G)$:
  - Let $u \in V(P)$ be incident with edges not in P. Consider $H = G[E(G) - E(P)]$.
  - $\forall x \in V(P) : \delta(x)$ is even $\Rightarrow \forall x \in V(H) : \delta(x)$ is even.
  - Let u lie in component $H' \Rightarrow$ construct similar largest trail P'
  - $P \leftarrow P \cup P'$ and repeat until $E(P) = E(G)$.

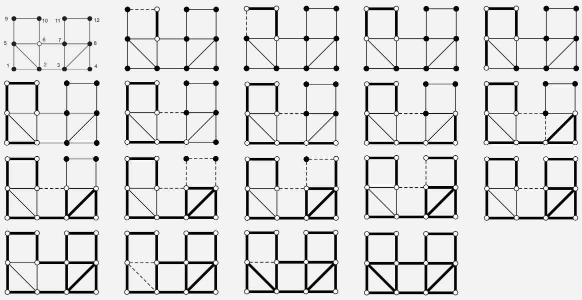
Fleury's algorithm

Algorithm (Fleury)

Consider an Eulerian graph G.

1. *Choose an arbitrary vertex $v_0 \in V(G)$ and set $W_0 = v_0$.*
2. *Assume that we have constructed a trail $W_k = [v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k]$. Choose an edge $e_{k+1} = \langle v_k, v_{k+1} \rangle$ from $E(G) \backslash E(W_k)$ such that, preferably, e_{k+1} is not a cut edge of the induced subgraph $G_k = G - E(W_k)$.*
3. *We now have a trail W_{k+1}. If there is no edge $e_{k+2} = \langle v_{k+1}, v_{k+2} \rangle$ to select from $E(G) \backslash E(W_{k+1})$, stop. Otherwise, repeat the previous step.*

Fleury's algorithm



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Chinese postman problem

Problem as a graph

Model city plan as a weighted graph:

- junction as a vertex
- street as edge, length represented by weight

Find a closed walk with minimal total weight.

Observation

We need to possibly make G Eulerian first by adding edges leading to G^* such that $\sum_{e \in E(G^*) \setminus E(G)} w(e)$ is minimal.

Question

Why may this be so difficult?

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Postman: example

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Postman: algorithm

Consider a weighted, connected graph G with odd-degree vertices $V_{odd} = \{v_1, \ldots, v_{2k}\}$ where $k \geq 1$.

1. For each pair of distinct odd-degree vertices v_i and v_j, find a minimum-weight (v_i, v_j)-path $P_{i,j}$.
2. Construct a weighted complete graph on $2k$ vertices in which vertex v_i and v_j are joined by an edge having weight $w(P_{i,j})$.
3. Find the set E of k edges $e_1, \ldots, e_k$ such that $\sum w(e_i)$ is minimal and no two edges are incident with the same vertex.
4. For each edge $e \in E$, with $e = \langle v_i, v_j \rangle$, duplicate the edges of $P_{i,j}$ in graph G.

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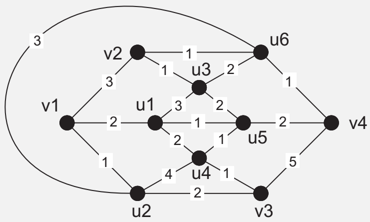
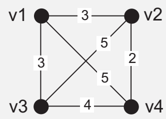
Postman: algorithm example





$P_{1,2} = [v_1, v_2]$ (weight: 3)
$P_{1,3} = [v_1, u_2, v_3]$ (weight: 3)
$P_{1,4} = [v_1, u_1, u_5, v_4]$ (weight: 5)
$P_{2,3} = [v_2, u_3, u_5, u_4, v_3]$ (weight: 5)
$P_{2,4} = [v_2, u_6, v_4]$ (weight: 2)
$P_{3,4} = [v_3, u_4, u_5, v_4]$ (weight: 4)

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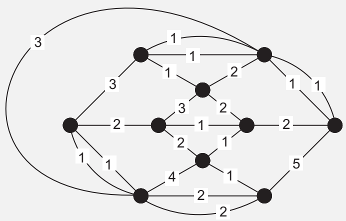
Postman: algorithm example



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Hamilton cycles

Definition

A Hamilton path of a connected graph G is a path that contains every vertex of G. A Hamilton cycle is a cycle containing every vertex of G. G is called Hamiltonian if it has a Hamilton cycle.

Important note

There is no known **efficient** algorithm to determine whether a graph is Hamiltonian. Yet, finding Hamilton cycles is important: **Traveling Salesman Problem** (**TSP**).

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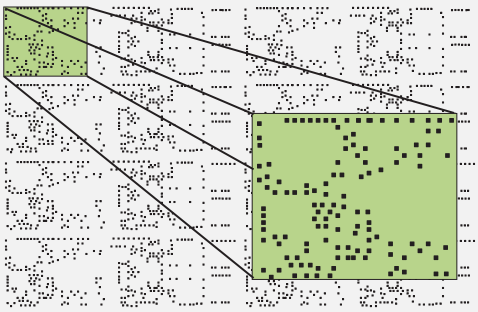
TSP: Example

Drilling holes: Consider a board for electrical circuits. To fasten the components, we need to drill holes. **Issue:** Which track should the drilling machine follow?



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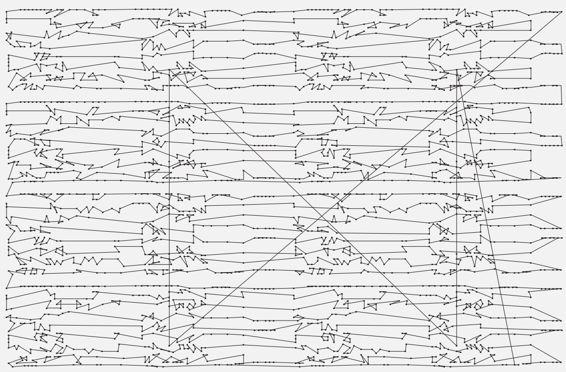
TSP: Example



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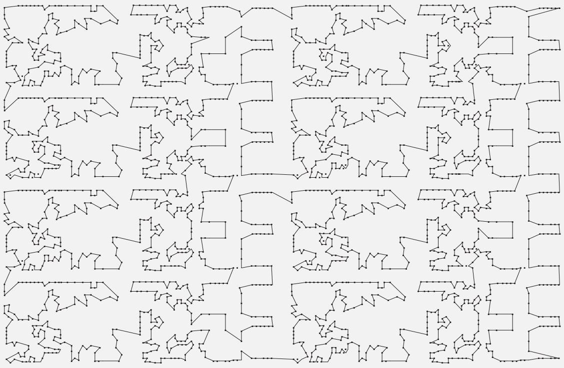
TSP: Example



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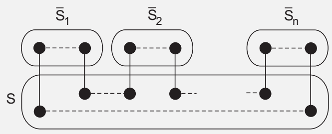
Some formal properties

Theorem

G Hamiltonian $\Rightarrow \forall S \subset V(G), S \neq \emptyset : \omega(G-S) \leq |S|$.

Proof

- Let C be a Hamilton cycle $\Rightarrow$ every vertex is visited exactly once $\Rightarrow \omega(C-S) \leq |S|$.
- $V(C) = V(G) \Rightarrow \omega(G-S) \leq \omega(C-S)$.



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Some formal properties: Dirac

Theorem (Dirac)

G is simple with $n \geq 3$ *vertices and* $\forall v : \delta(v) \geq n/2 \Rightarrow$ *G is Hamiltonian.*

Proof: by induction

- For $n = 3$ vertices: trivial. Assume the theorem has been proven correct for graphs with $k \geq 3$ vertices.
- Let G have $k+1$ vertices, constructed from any graph G^* with k vertices, by adding a vertex u and joining u to at least $(k+1)/2$ other vertices.
- Let $C^* = [v_1, v_2, \ldots, v_k]$ be a Hamilton cycle in G^*.
- Vertex u is joined to at least $(k+1)/2$ vertices from C^* $\Rightarrow$ there is at least a pair v_i and v_{i+1} that are adjacent in C^*
- Construct a new cycle $C = [v_1, \ldots, v_i, u, v_{i+1}, v_k]$

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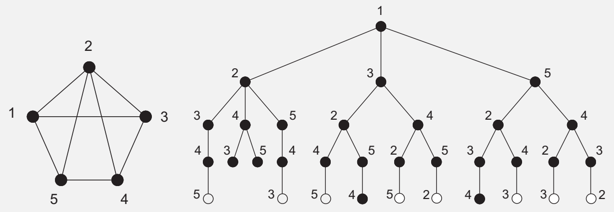
Finding Hamilton cycles

Brute force: Select a vertex v, and explore all possible Hamilton paths originating from v, and check whether they can be expanded to a cycle:



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Posa: applying rotational transformations

Algorithm (Posa)

Randomly select $u \in V(G)$, forming the starting point of path P. Let $last(P) = u$ denote the current end point of P.

1. *Randomly select $v \in N(last(P))$, such that*
   1. *Preferably, $v \notin V(P)$*
   2. *If $v \in V(P) \Rightarrow v$ has not been previously selected as neighbor of an end point before.*

   If no such vertex exists, stop.
2. *If $v \notin V(P)$, set $P \leftarrow P + \langle last(P), v\rangle$.*

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Posa: applying rotational transformations

Algorithm (Posa - cntd)

3. *If $v \in V(P)$, apply a **rotational transformation of** P using edge $\langle last(P), v\rangle$:*

   

   leading to P^*. If $last(P^*)$ has not yet been end point for paths of the current length, $P \leftarrow P^*$.
4. *$V(P) = V(G)$ and $\langle u, last(P)\rangle \in E(G) \Rightarrow$ found a Hamilton cycle. Otherwise, continue with step 1.*

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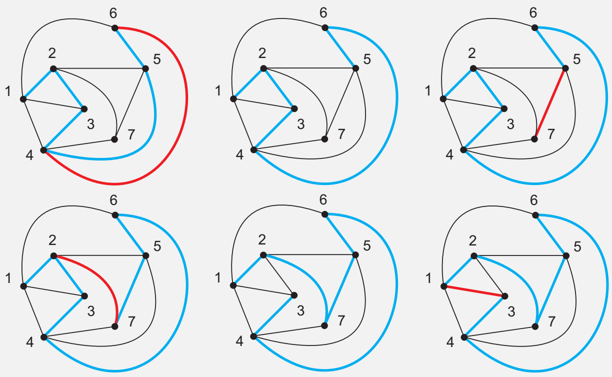
Posa example



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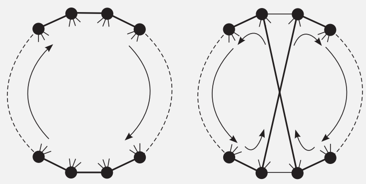
Optimal Hamilton cycle

Basic idea

We want to find a Hamilton cycle with **minimal weight** $\Rightarrow$ extend graph to a complete one in which distance between two vertices reflects real-world distance.



(a) (b)

(a) Start with an arbitrary cycle

(b) If swapping edges improve weight $\Rightarrow$ better cycle

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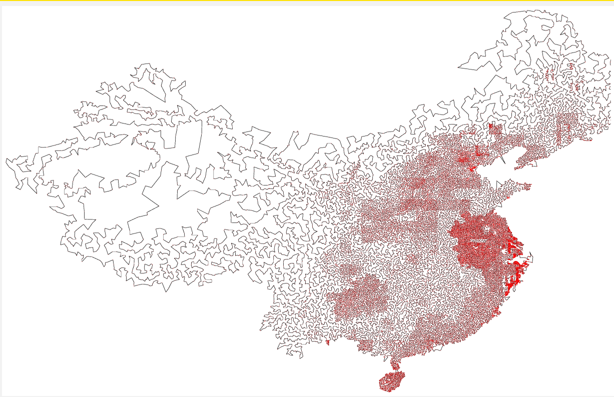
Hamilton example: China



71,000 cities, 4,566,563 edges $\leq$ 0.024% longer than optimal one.

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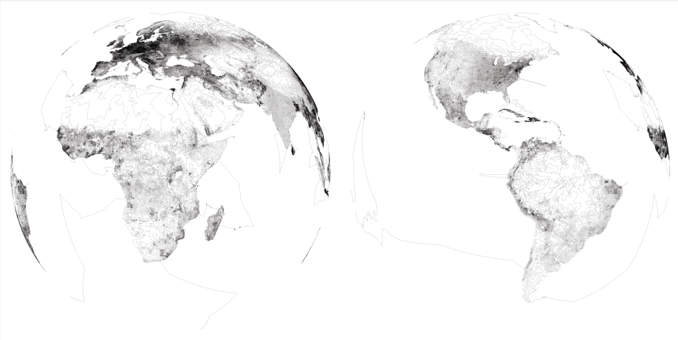
Hamilton example: The world



1,904,711 cities, 7,516,353,779 edges $\leq$ 0.076% longer than optimal one.

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