Graph Theory and Complex Networks: An Introduction

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Chapter 03: Extensions

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Contents

Directed graph

Idea: extend graphs by letting edges have an explicit direction:

- Representing one-way streets in a street plan
- Expressing asymmetry in social relationships (Alice likes Bob: $A \rightarrow B$)
- Expressing asymmetry in communication networks

Definition

A directed graph or digraph D is a tuple (V, A) of vertices V, and a collection of arcs A where each arc $a = \langle \overrightarrow{u,v} \rangle$ joins a vertex (tail) $u \in V$ to another (not necessarily distinct) vertex (head) v.

Extensions 3.1 Directed graphs

Basic properties

Definition

For a vertex v of digraph D, the number of arcs with head v is called the indegree $\delta_{in}(v)$ of v. The outdegree $\delta_{out}(v)$ is the number of arcs having v as their tail.

Theorem

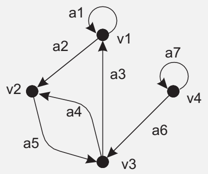
$\forall D : \sum_{v \in V(D)} \delta_{in}(v) = \sum_{v \in V(D)} \delta_{out}(v) = |A(D)|$

Proof

- Every arc in D has exactly one head and one tail.
- $\sum_{v \in V(D)} \delta_{in}(v)$ is the same as counting all arc heads
- $\sum_{v \in V(D)} \delta_{out}(v)$ is the same as counting all tails
- Both are equal to the total number of arcs.

Extensions 3.1 Directed graphs

Adjacency matrix



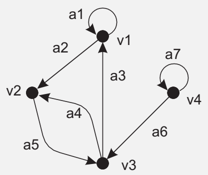
	v_1	v_2	v_3	v_4	$\sum$
v_1	1	1	0	0	2
v_2	0	0	1	0	1
v_3	1	1	0	0	2
v_4	0	0	1	1	2
$\sum$	2	2	2	1	7

Observations

- Adjacency matrix is *not* necessarily symmetric: in general, $\mathbf{A}[i,j] \neq \mathbf{A}[j,i]$.
- A digraph D is strict iff $\mathbf{A}[i,j] \leq 1$ and $\mathbf{A}[i,i] = 0$.
- $\forall v_i : \sum_j \mathbf{A}[i,j] = \delta_{out}(v_i)$ and $\sum_j \mathbf{A}[j,i] = \delta_{in}(v_i)$.

Extensions 3.1 Directed graphs

Incidence matrix



	a_1	a_2	a_3	a_4	a_5	a_6	a_7
v_1	0	1	-1	0	0	0	0
v_2	0	-1	0	-1	1	0	0
v_3	0	0	1	1	-1	-1	0
v_4	0	0	0	0	0	1	0

$$\mathbf{M}[i,j] = \begin{cases} 1 & \text{if vertex } v_i \text{ is the tail of arc } a_j \\ -1 & \text{if vertex } v_i \text{ is the head of arc } a_j \\ 0 & \textit{otherwise} \end{cases}$$

Observation

Incidence matrices for digraphs cannot capture loops, making these matrices being used less often compared to undirected graphs.

Extensions 3.1 Directed graphs

Connectivity

Definition

A directed $(\mathbf{v_0}, \mathbf{v_k})$-walk is an alternating sequence $[v_0, a_0, v_1, a_1, \ldots, v_{k-1}, a_{k-1}, v_k]$ with $a_i = \langle \overrightarrow{v_i, v_{i+1}} \rangle$.

- A directed trail is a directed walk with distinct arcs.
- a directed path is a directed trail with distinct vertices.
- a directed cycle is a directed trail with distinct vertices except for $v_0 = v_k$.

Definition

D is strongly connected if there exists a directed path between every pair of distinct vertices from D. D is weakly connected if its underlying (undirected) graph is connected.

Extensions 3.1 Directed graphs

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Reachability

Definition

Vertex v is reachable from vertex u if there exists a directed (u, v)-path.

Algorithm (Reachable vertices)

$R_t(u)$ is set of reachable vertices from u found after t steps.
$N_{out}(v)$ is out-neighbors of v: $N_{out}(v) = \{w \in V(D) | \exists \langle \overrightarrow{v, w} \rangle \in A(D)\}$.

1. *Set $t \leftarrow 0$ and $R_0(u) \leftarrow \{u\}$.*
2. *Construct the set $R_{t+1}(u) \leftarrow R_t(u) \cup \left(\bigcup_{v \in R_t(u)} N_{out}(v) \right)$.*
3. *If $R_{t+1}(u) = R_t(u)$, stop: $R(u) \leftarrow R_t(u)$. Otherwise, increment t and repeat the previous step.*

Extensions 3.1 Directed graphs

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Strongly connected orientations

Note

An orientation $D(G)$ of an undirected graph G is a directed graph in which edge from G has been assigned a direction.

Question

Given G, how many orientations can you construct?

Theorem

There exists an orientation $D(G)$ for a connected undirected graph G that is strongly connected if and only if $\lambda(G) \geq 2$.

Proof: Strongly connected $\Rightarrow \lambda(G) \geq 2$

By contradiction: assume that $\lambda(G) = 1$.

Extensions 3.1 Directed graphs

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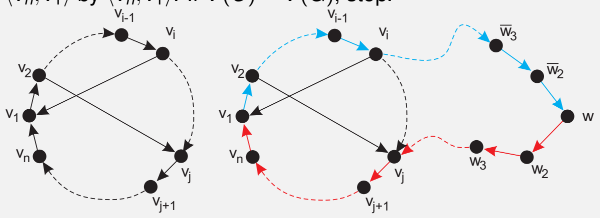
Proof: $\lambda(G) \geq 2 \Rightarrow$ exists strongly conn. orientation

- $\lambda(G) \geq 2 \Rightarrow$ every edge lies on a cycle.
- $C = [v_1, v_2, \ldots, v_n, v_1] \Rightarrow \langle v_i, v_{i+1} \rangle$ is replaced with arc $\langle \overrightarrow{v_i, v_{i+1}} \rangle$; $\langle v_n, v_1 \rangle$ by $\langle \overrightarrow{v_n, v_1} \rangle$. If $V(C) = V(G)$, stop.



- $V(C) \neq V(G)$. Let $w \notin V(C)$. $\lambda(G) \geq 2 \Rightarrow$ there are two edge-independent (w, v_1)-paths P_1 and P_2. Set orientation.
- Repeat until $W = V(C) \cup V(P_1) \cup V(P_2) = V(G)$

Extensions 3.1 Directed graphs

Weighted graphs

Definition

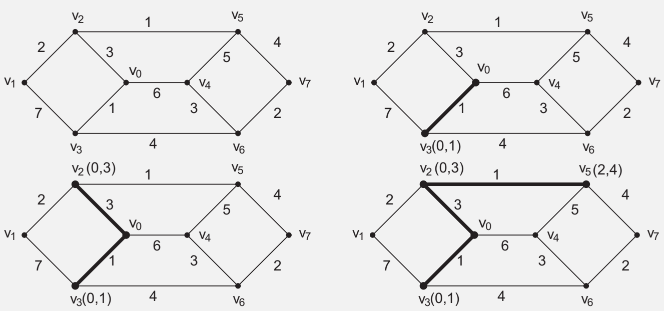
In a weighted graph G each edge e has an associated real-valued weight $w(e) < \infty$. For $H \subseteq G$, $w(H) = \sum_{e \in E(H)} w(e)$.

Important application: Finding the shortest path in a graph. **Basic idea**:

- Start with a set $S = \{v_0\}$, and add vertex closest to v_0.
- Expand S by adding vertex closest to v_0 through one of the vertices in S.
- Stop when there are no more vertices left.

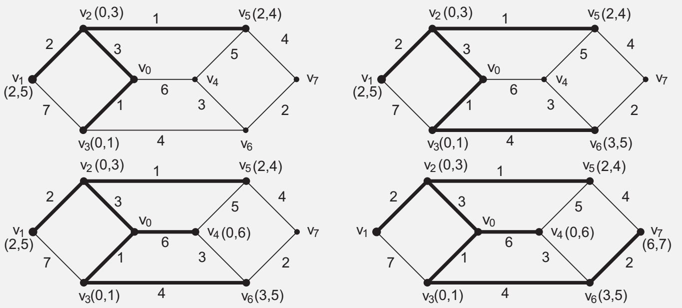
Extensions 3.2 Weighted graphs

Dijkstra's algorithm



Extensions 3.2 Weighted graphs

Dijkstra's algorithm



Extensions 3.2 Weighted graphs

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Edge colorings

Basic idea

Assign colors to edges such that two edges incident to the same vertex have different colors:
$\forall \langle u, v \rangle, \langle v, w \rangle \in E(G) : col(\langle u, v \rangle) \neq col(\langle v, w \rangle)$.

Application

Consider n storage devices, but that we need to move data between devices (e.g., to balance the load).

- Represent each storage device by a vertex.
- Divide all data into equally sized data blocks.
- If data block b needs to be moved from device i to j: add arc $\langle \overrightarrow{i, j} \rangle$.
  Note: we may have multiple arcs from i to j.

Extensions 3.3 Colorings

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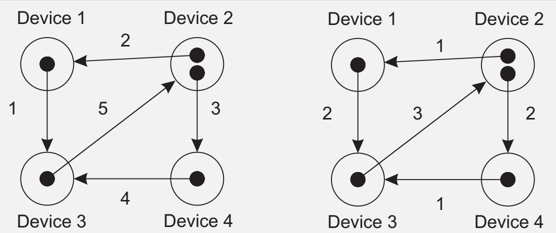
Extensions 3.3 Colorings

Edge colorings: example

Problem

Can we devise a **migration schedule** that does the job as quickly as possible, under the assumption that each device can move/accept only one block at a time?



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Edge colorings: formalities

Definition

G, connected and loopless, is k-edge colorable if $E(G)$ can be partitioned into k disjoint sets $E_1, \ldots, E_k$ such that $\forall E_i : e_1, e_2 \in E_i \Rightarrow e_1, e_2$ are not incident with the same vertex.

Edge chromatic number: minimal k for which G is k-edge colorable: $\chi'(G)$.

Theorem (Vizing)

For any simple graph G, either $\chi'(G) = \Delta(G)$ *or* $\chi'(G) = \Delta(G) + 1$, *with* $\Delta(G) = max_{v \in V(G)} \delta(v)$

Note

For all graphs we have $\chi'(G) \geq \Delta(G)$

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Vertex colorings

Definition

G, simple and connected, is k-vertex colorable if $V(G)$ can be partitioned into k disjoint sets $V_1, \ldots, V_k$ such that $\forall V_i, \forall x, y \in V_i : \langle x, y \rangle \notin E(G)$.

Chromatic number: minimal k for which G is k-vertex colorable: $\chi(G)$.

Problem

Finding $\chi(G)$ is a notoriously difficult problem: no efficient general solution exists, meaning we need to essentially try all possible combinations.

Extensions 3.3 Colorings

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Finding $\chi(G)$

Theorem

For any (simple, connected) graph G: $\chi(G) \leq \Delta(G) + 1$.

Proof by induction on number of vertices n

- $n = 1$: trivial as $\chi = 1$ and $\Delta = 0$.
- Assume OK for $k > 0$ and consider G with $|V(G)| = k + 1$.
- Consider $v \in V$ with $\delta(v) = \Delta(G)$. $G^* = G - v \Rightarrow$ exists c-vertex coloring C^* of G^* with $\chi(G^*) = c \leq \Delta(G^*) + 1$.
- $\Delta(G) = \Delta(G^*) \Rightarrow$ worst case $c = \Delta(G^*) + 1$. $|N(v)| = \Delta(G) = c - 1 \Rightarrow$ there is a color left over that we can use for v.
- $\Delta(G) > \Delta(G^*) \Rightarrow$ introduce new color for v and at worst $\chi(G) = \chi(G^*) + 1 \leq \Delta(G^*) + 2 \leq \Delta(G) + 1$.

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Coloring planar graphs

Theorem

For any planar graph G, $\chi(G) \leq 4$.

Observation

If this theorem holds, we should be able to color any map with only four different colors.

Problem

- Conjectured in 1852 and specific cases proved to hold.
- Only in 1976 the theorem was proved to be true, but...
- A computer program was needed:
  - Split problem into 2000 different cases
  - Write a program for each case separately
  - Were the programs correct?

Extensions 3.3 Colorings

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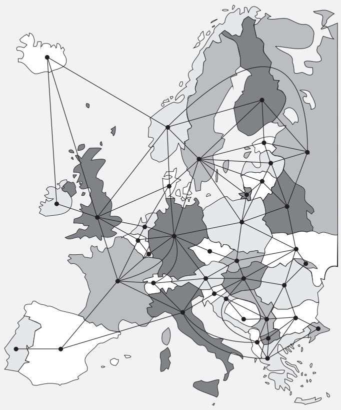
Map coloring



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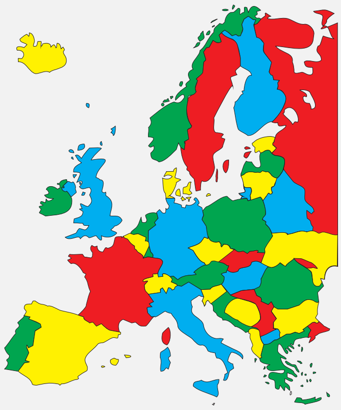
Map coloring



Extensions 3.3 Colorings

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Simpler bounds for $\chi(G)$

Theorem

Every planar graph has a vertex v with $\delta(v) \leq 5$.

Proof

- Consider only $n \geq 7$ vertices (otherwise trivial);
- $m = |E(G)| \Rightarrow \sum_{v \in V(G)} \delta(v) = 2m$.
- Assume no vertex exists with $\delta(v) \leq 5 \Rightarrow 6n \leq 2m$.
- G planar $\Rightarrow m \leq 3n - 6 \Rightarrow 6n \leq 6n - 12$. Contradiction.

Simpler bounds for $\chi(G)$

Theorem

For any planar graph G, $\chi(G) \leq 5$.

Proof by induction on number of vertices n

- $n = 1$: obviously true. Assume correct for all graphs with $k > 1$ vertices.
- Consider G with $k + 1$ vertices. Let v have $\delta(v) \leq 5$.
- $G^* = G - v$ has k vertices $\Rightarrow$ exists 5 coloring with colors $c_1, \ldots, c_5$.
- Not all colors used in $N(v) \Rightarrow$ assign unused color to v. Done.

Simpler bounds for $\chi(G)$

Proof cnt'd: assume all colors used for $N(v) \Rightarrow \delta(v) = 5$

Idea: Rearrange the colors in $N(v) = \{v_1, v_2, \ldots, v_5\}$. Let $col(v_i) = c_i$.

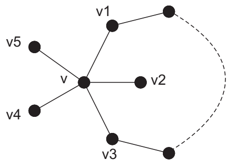
Assume no (v_1, v_3)-path in G^* with only c_1, c_3: Consider (v_1, w)-paths in G^* colored with only c_1, c_3

- For the induced subgraph H, we know that $v_3 \notin V(H)$
- Also: $N(v_3) \cap V(H) = \emptyset$.

Solution: interchange c_1 and c_3 in $H \Rightarrow$ use c_1 for v.

Simpler bounds for $\chi(G)$

Proof cnt'd: assume all colors used for $N(v) \Rightarrow \delta(v) = 5$



Assume there exists (v_1, v_3)-path P in G^* with only c_1, c_3: Consider cycle $C = [v_3, v, v_1, P]$. C encloses v_2, or otherwise v_4 and $v_5 \Rightarrow$ no (v_2, v_4)-path with only colors c_2, c_4. Consider all (v_2, w)-paths with only colors c_2, c_4. Induce subgraph H' of G^*.

Solution: interchange colors c_2 and c_4 in $H' \Rightarrow$ use c_2 for v.

Extensions 3.3 Colorings

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