Graph Theory and Complex Networks: An Introduction

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Chapter 02: Foundations

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Graph: definition

Definition

A graph G is a tuple (V,E) of vertices V and a collection of edges E. Each edge $e \in E$ is said to connect two vertices $u,v \in V$, and is denoted as $e = \langle u,v \rangle$.
Notations: $V(G)$, $E(G)$.

Definition

The complement $\overline{G}$ of a graph G, has the same vertex set as G, but $e \in E(\overline{G})$ *if and only if* $e \notin E(G)$.

Definition

For any graph G and vertex $v \in V(G)$, the neighbor set $N(v)$ of v is the set of vertices (other than v) adjacent to v:

$$N(v) = \{w \in V(G) \mid v \neq w, \langle v,w \rangle \in E(G)\}$$

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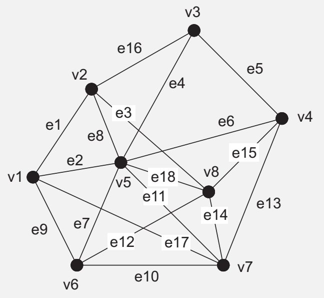
Graph: Example



$V(G) = \{v_1, \ldots, v_8\}$
$E(G) = \{e_1, \ldots, e_{18}\}$

$e_1 = \langle v_1, v_2 \rangle$	$e_{10} = \langle v_6, v_7 \rangle$
$e_2 = \langle v_1, v_5 \rangle$	$e_{11} = \langle v_5, v_7 \rangle$
$e_3 = \langle v_2, v_8 \rangle$	$e_{12} = \langle v_6, v_8 \rangle$
$e_4 = \langle v_3, v_5 \rangle$	$e_{13} = \langle v_4, v_7 \rangle$
$e_5 = \langle v_3, v_4 \rangle$	$e_{14} = \langle v_7, v_8 \rangle$
$e_6 = \langle v_4, v_5 \rangle$	$e_{15} = \langle v_4, v_8 \rangle$
$e_7 = \langle v_5, v_6 \rangle$	$e_{16} = \langle v_2, v_3 \rangle$
$e_8 = \langle v_2, v_5 \rangle$	$e_{17} = \langle v_1, v_7 \rangle$
$e_9 = \langle v_1, v_6 \rangle$	$e_{18} = \langle v_5, v_8 \rangle$

Question

What is the neighborset of v_6?

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Vertex degree

Definition

The number of edges incident with a vertex v is called the degree of v, denoted as $\delta(v)$. Loops, i.e., edges joining a vertex with itself, are counted twice.

Theorem

For all graphs G, $\sum_{v \in V(G)} \delta(v)$ is $2 \cdot |E(G)|$.

Proof

When we count the edges of a graph G by enumerating the edges incident with each vertex of G, we are counting each edge exactly twice.

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Degree sequence

Definition

An (ordered) degree sequence is an (ordered) list of the degrees of the vertices of a graph. A degree sequence is graphic if there is a (simple) graph with that sequence.

Theorem (Havel-Hakimi)

An ordered degree sequence $\mathbf{s} = [k, d_1, d_2, \ldots, d_{n-1}]$ is graphic, if and only if $\mathbf{s}^* = [d_1 - 1, d_2 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_{n-1}]$ is also graphic. (We assume $k \geq d_i \geq d_{i+1}$.)

Note

Length $\mathbf{s} = n$, but length $\mathbf{s}^* = n - 1$.

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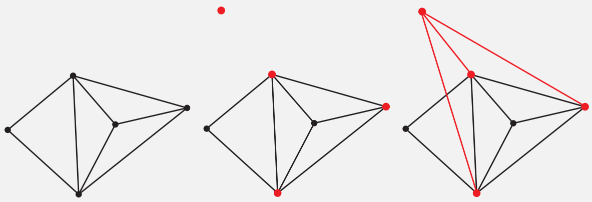
$\mathbf{s}^* \Rightarrow \mathbf{s}$: Example

Take $k = 3$ and consider graph with sequence $[4,4,3,3,2]$. Create graph with sequence $[3,5,5,4,3,2] \equiv [5,5,4,3,3,2]$:



1. Starting condition
2. Add a vertex v with degree $\delta(v) = k$
3. Connect v to k vertices with highest degrees.

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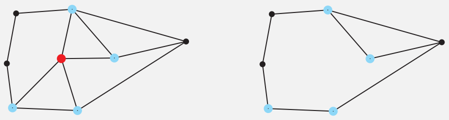
$\mathbf{s} \Rightarrow \mathbf{s}^*$: Example

Consider the following graph with sequence $[4,4,3,3,3,3,2,2]$. Let $\delta(u) = 4$ (in red) and consider $V = \{v_1, v_2, v_3, v_4\}$ as next highest degrees (in blue), and $W = \{w_1, w_2, w_3\}$ the rest (in black).



1. Starting condition
2. Remove u. Because u is connected only to vertices from V, we know that $s^* = [3,2,2,2,3,2,2] = [3,3,2,2,2,2,2]$

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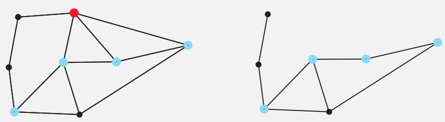
$\mathbf{s} \Rightarrow \mathbf{s}^*$: Example

Consider the following graph with sequence $[4,4,3,3,3,3,2,2]$. Let $\delta(u) = 4$ (in red) and consider $V = \{v_1, v_2, v_3, v_4\}$ as next highest degrees (in blue), and $W = \{w_1, w_2, w_3\}$ the rest (in black).



1. Starting condition
2. Remove u. Because u is **not** connected only to vertices from V, we have a problem: $s^* = [3,3,3,2,2,2,1]$.

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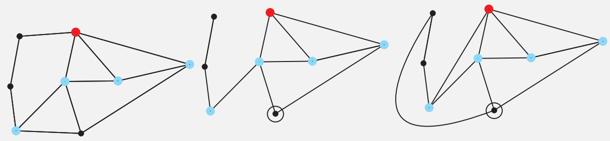
$\mathbf{s} \Rightarrow \mathbf{s}^*$: Example



1. Problem: u is linked to a w but not to a v_j, with $\delta(w) < \delta(v_j)$. But because $\delta(w) < \delta(v_j)$, there exists x adjacent to v_j but not to w.
2. Remove $\langle u, w\rangle$ and $\langle v_j, x\rangle$.
3. Add $\langle x, w\rangle$ and $\langle u, v_j\rangle$

Question

What should we do if u was linked to a w with $\delta(w) = \delta(v_j)$?

Foundations 2.1 Formalities

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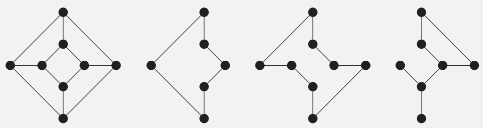
Subgraphs

Definition

H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ such that for all $e \in E(H)$ with $e = \langle u, v\rangle : u, v \in V(H)$.



Definition

The subgraph induced by $V^* \subseteq V(G)$ has vertex set V^* and edge set $\{\langle v, w\rangle \in E(G) | v, w \in V^*\}$. Denoted as $H = G[V^*]$. The subgraph induced by $E^* \subseteq E(G)$ has vertex set $V(G)$ and edge set E^*. Denoted as $H = G[E^*]$.

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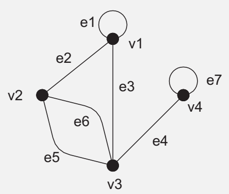
Adjacency matrix



	v_1	v_2	v_3	v_4
v_1	2	1	1	0
v_2	1	0	2	0
v_3	1	2	0	1
v_4	0	0	1	2

Observations

- Adjacency matrix is *symmetric*: $\mathbf{A}[i,j] = \mathbf{A}[j,i]$.
- G is simple $\Leftrightarrow \mathbf{A}[i,j] \leq 1$ and $\mathbf{A}[i,i] = 0$.
- $\forall v_i$: $\sum_{j=1}^{n} \mathbf{A}[i,j] = \delta(v_i)$.

Foundations 2.2 Graph representations

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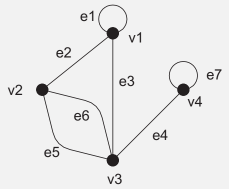
Incidence matrix



	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	2	1	1	0	0	0	0
v_2	0	1	0	0	1	1	0
v_3	0	0	1	1	1	1	0
v_4	0	0	0	1	0	0	2

Observations

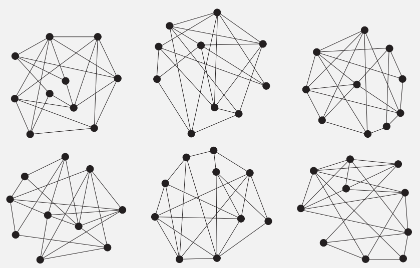
- G is simple only if $\mathbf{M}[i,j] \leq 1$
- $\forall v_i$: $\sum_{j=1}^{m} \mathbf{M}[i,j] = \delta(v_i)$.
- $\forall e_j$: $\sum_{i=1}^{n} \mathbf{M}[i,j] = 2$.

Foundations 2.2 Graph representations

Graph isomorphism

Definition

G_1 and G_2 are isomorphic if there exists a one-to-one mapping $\phi : V_1 \rightarrow V_2$ such that for each edge $e_1 \in E_1$ with $e_1 = \langle v, u \rangle$ there is a unique edge $e_2 \in E_2$ with $e_2 = \langle \phi(v), \phi(u) \rangle$.



Foundations 2.2 Graph representations

Connectivity: definitions

Definition

A $(\mathbf{v_0}, \mathbf{v_k})$-walk is a sequence $[v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k]$ with $e_i = \langle v_{i-1}, v_i \rangle$. A trail is a walk with distinct edges; a path is a trail with distinct vertices. A cycle is a trail with distinct vertices except $v_0 = v_k$.

Definition

Vertices $u \neq v$ in G are connected if there is a $(u,v) - path$ in G. G is connected if all pairs of distinct vertices are connected.

Definition

$H \subseteq G$ is a component of G if H is connected and not contained in a connected subgraph of G with more vertices or edges. The number of components of G is $\omega(G)$.

Foundations 2.3 Connectivity

Connectivity and robustness

Important

Connectivity indicates whether all nodes in a network can be reached from any other node.

Example

Communication networks, like the Internet, require to be connected, and have been designed to stay connected, even when under attack.

Definition

For a graph G let $V^* \subset V(G)$ and $E^* \subset E(G)$. If $\omega(G-V^*) > \omega(G)$ then V^* is called a vertex cut. If $\omega(G-E^*) > \omega(G)$ then E^* is called an edge cut.

Minimal cuts

Note

For reasons of robustness, we're interested in finding the minimal number of vertices or edges to remove before a graph falls apart.

Notations

- $\kappa(G)$ is the size of a minimal vertex cut for G
- $\lambda(G)$ is the size of a minimal edge cut

Theorem

$\kappa(G) \leq \lambda(G) \leq \min_{v \in V(G)}\{\delta(v)\}$

$\kappa(G) \leq \lambda(G) \leq \min_{v \in V(G)}\{\delta(v)\}$

$\lambda(G) \leq \min_{v \in V(G)}\{\delta(v)\}$ Let u have minimal degree $\Rightarrow$ remove the edges incident with it and u becomes isolated.

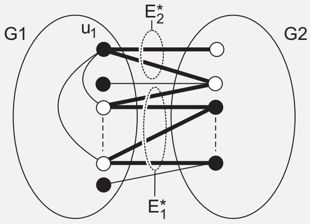
$\kappa(G) \leq \lambda(G)$ Let $E^* = \{\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_k, v_k\rangle\}$ be an edge cut, with $k = \lambda(G) \Rightarrow G - E^*$ falls into exactly two components G_1 and G_2 (**why?**).

- Assume there exists $u \in V(G_1) \backslash \{u_1, \ldots, u_k\}$. This means that $\{u_1, \ldots, u_k\}$ is a vertex cut $\Rightarrow \kappa(G) \leq k$.

$\kappa(G) \leq \lambda(G) \leq \min_{v \in V(G)}\{\delta(v)\}$ (cnt'd)

Otherwise, assume $V(G_1) = \{u_1, \ldots, u_k\}$ and consider vertex u_1.

- u_1 is adjacent to d_1 vertices $N_1(u_1)$ from $V(G_1)$ and d_2 vertices $N_2(u_1)$ from $V(G_2)$.
- Each $u_i \in N_1(u_1)$ is adjacent to a vertex from $V(G_2)$.
- Let $E_1^* = \{\langle u, v\rangle \in E^* | u \in N_1(u_1), v \in V(G_2)\}$ $E_2^* = \{\langle u_1, v\rangle \in E^* | v \in N_2(u_1)\}$
- $d_1 + d_2 \leq |E_1^*| + d_2 \leq |E_1^*| + |E_2^*| \leq |E^*| = \lambda(G)$.
- $N_1(u_1) \cup N_2(u_1)$ is a vertex cut with $d_1 + d_2$ vertices.



Foundations 2.3 Connectivity

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What does it take to be connected?

Definition

If $\kappa(G) \geq k$ for some k, then G is called k-connected.

Note

G is k-connected $\Rightarrow \forall v : \delta(v) \geq k$

Issue

Can we construct a k-connected graph $H_{k,n}$ with n vertices and a minimal number of edges?

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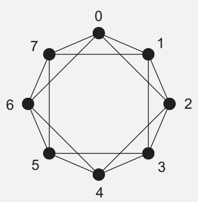
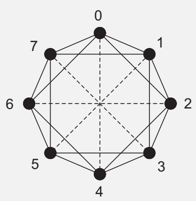
Harary graphs

k is even: Organize vertices $V = \{0, 1, \ldots, n-1\}$ into a "circle." Connect vertex i to its $k/2$ left-hand (clockwise) neighbors and to its $k/2$ right-hand (counter clockwise) neighbors.

k is odd, n is even: Construct $H_{k-1,n}$ and add edges $\langle 0, \frac{n}{2}\rangle, \langle 1, 1 + \frac{n}{2}\rangle, \ldots, \langle \frac{n-2}{2}, n-1\rangle$.



$H_{4,8}$



$H_{5,8}$

Foundations 2.3 Connectivity

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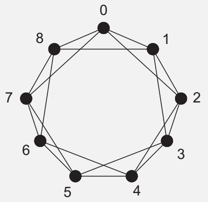
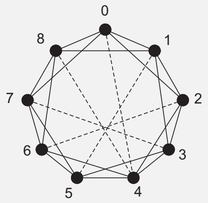
Harary graphs

k is odd, n is odd: Construct $H_{k-1,n}$ and add edges $\langle 0, \frac{n-1}{2} \rangle, \langle 1, 1+\frac{n-1}{2} \rangle, \dots, \langle \frac{n-1}{2}, n-1 \rangle$.



$H_{4,9}$



$H_{5,9}$

Foundations 2.3 Connectivity

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Menger's theorem

Definition

Let $\mathscr{P}(u,v)$ be a collection of paths between vertices u and v.
Vertex independent: $\forall P,Q \in \mathscr{P}(u,v) : V(P) \cap V(Q) = \{u,v\}$.
Edge independent: $\forall P,Q \in \mathscr{P}(u,v) : E(P) \cap E(Q) = \emptyset$.

Theorem (Menger)

Let G be a graph with two nonadjacent vertices u and v. The minimum number of vertices in a vertex cut that disconnects u and v is equal to the maximum number of pairwise vertex-independent paths between u to v. The minimum number of edges in an edge cut that disconnects u and v, is equal to the maximum number of pairwise edge-independent paths betweeen u and v.

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Menger's theorem

Mathematical language

Menger's theorem should be read carefully: it mentions pairwise independent paths. In this case, the adjective pairwise is used to make clear that we should always consider pairs of paths when considering independence. And indeed, this makes sense when you would consider trying to count the number of independent paths: being an independent path can only be relative to another path.

To complete the story, also note that the theorem is all about counting the number of (u,v)-paths, and not the number of pairs of such paths. In other words, pairwise is an adjective to independent, and not to paths.

Foundations 2.3 Connectivity

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Corollaries

Corollary

- *G is k-connected iff any two distinct vertices are connected by at least k vertex-independent paths.*
- *G is k-edge connected iff any two distinct vertices are connected by at least k edge-independent paths.*

Corollary

Each edge of a 2-edge-connected graph lies on a cycle.

Foundations 2.3 Connectivity

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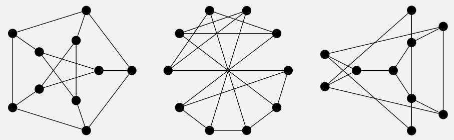
Drawing graphs

Observation

It is important to see how you draw a graph, that is, to consider its **graph embedding**.



Foundations 2.4 Drawing graphs

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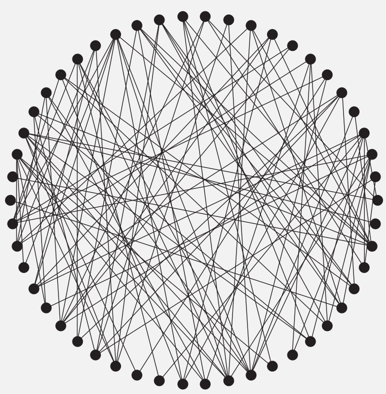
Circular embedding



Foundations 2.4 Drawing graphs

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Ranked embedding

Definition

G is bipartite if $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ such that $E(G) \subseteq \{\langle u_1, u_2 \rangle | u_1 \in V_1, u_2 \in V_2\}$.

1. Consider bipartite graph G and vertex $v \in V(G)$
2. Let $N_0^*(v) = \{v\}$
3. Let $N_k^*(v) = N_{k-1}^*(v) \cup \{x \in N(y) | y \in N_{k-1}^*(v)\}, k \geq 1$
4. $N_k(v) = N_k^*(v) - N_{k-1}^*(v)$
5. Draw vertices from $N_k(v)$ on the same vertical line, and vertices from $N_{k-1}(v)$ below (or above) those of $N_k(v)$.

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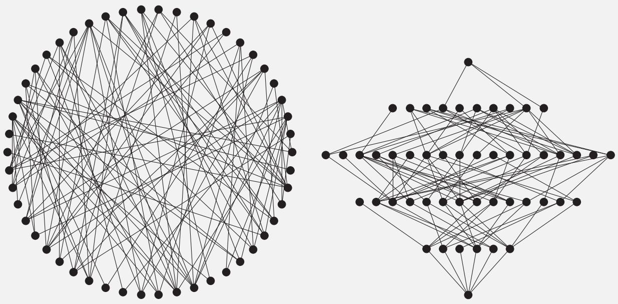
Ranked embedding



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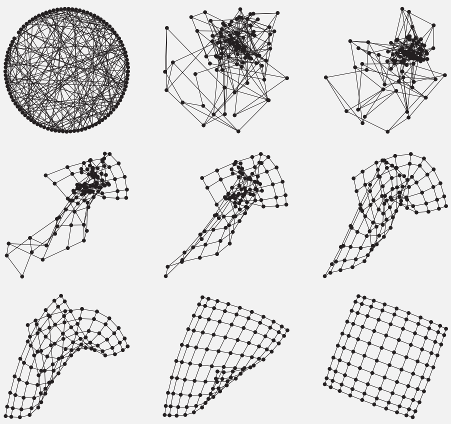
Spring embedding



Foundations 2.4 Drawing graphs

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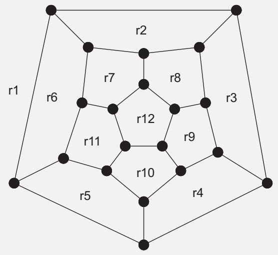
Planar graphs

Definition

A graph is planar if there exists an embedding in the 2D plane such that no two edges cross. A plane graph is a drawing of a planar graph such that no two edges intersect.



Theorem (Euler's formula)

For a plane graph with n vertices, m edges, and r regions: $n - m + r = 2$.

Foundations 2.4 Drawing graphs

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Planar graphs: properties

Theorem

For any connected simple planar graph with $n \geq 3$ vertices and m edges: $m \leq 3n - 6$

- Consider region f in a plane graph of G
- $\forall$ interior regions: $B(f)$ denotes number of edges enclosing f. Note: $B(f) \geq 3$.
- $n \geq 3 \Rightarrow$ exterior region bounded by at least 3 edges.
- r regions $\Rightarrow \sum B(f) \geq 3r$
- $\sum B(f)$ counts edges once or twice $\Rightarrow \sum B(f) \leq 2m$
- $3r \leq \sum B(f) \leq 2m \Rightarrow m = n + r - 2 \leq n + \frac{2}{3}m - 2 \Rightarrow m \leq 3n - 6$

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