An Introduction to

Graph Theory and Complex Networks

PROBLEMS

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Contents

Problems Chapter 2

Q 1: Give the adjacency matrix for each of the following graphs, and draw those graphs.

G1: $V = \{1, 2, 3, 4, 5, 6\}$ and
$E = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 3,5\rangle, \langle 3,6\rangle, \langle 4,5\rangle, \langle 4,6\rangle\}$

G2: $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 3,5\rangle\}$

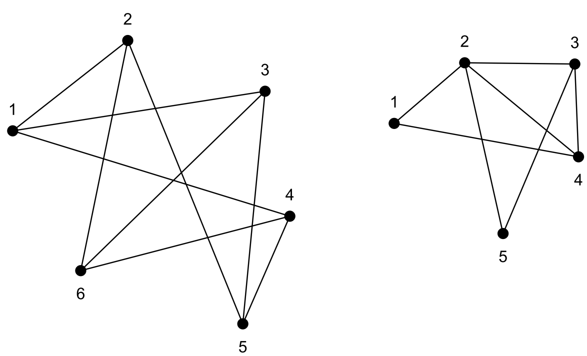
Q 2: Consider the following two graphs:

G1: $V = \{1, 2, 3, 4, 5, 6\}$ and
$E = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 3,5\rangle, \langle 3,6\rangle, \langle 4,5\rangle, \langle 4,6\rangle\}$

G2: $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 3,5\rangle\}$

For each graph, check whether it is (1) bipartite, (2) complete, (3) complete bipartite, (4) complete nonbaprtite.

Q 3: Draw the complement of the following two graphs:



Q 4: Prove that for any graph, the sum of its vertex degrees is even.

Q 5: Show that every simple graph has two vertices of the same degree.

Q 6: Show that if n people attend a party and some shake hands with others (but not with themselves), then at the end, there are at least two people who have shaken hands with the same number of people.

Q 7: Show that if every component of a graph is bipartite, then the graph is bipartite.

Q 8: Show that the complement of a bipartite graph need not to be a bipartite graph.

Q 9: Prove the following. Consider a list $\mathbf{s} = [d_1, d_2, \ldots, d_n]$ of n numbers in descending order. This list is graphic if and only if $\mathbf{s}^* = [d_1^*, d_2^*, \ldots, d_{n-1}^*]$ of $n-1$ numbers is graphic as well, where

$$d_i^* = \begin{cases} d_{i+1} - 1 & \text{for } i = 1, 2, \ldots, d_1 \\ d_{i+1} & \text{otherwise} \end{cases}$$

Q 10: Show that two graphs with the same degree sequence need not be isomorphic.

Q 11: Show that there is no simple graph with 12 vertices and 28 edges in which

(a) the degree of each vertex is either 3 or 4, or

(b) the degree of each vertex is either 3 or 6.

Q 12: Show that there is no simple graph with four vertices such that three vertices have degree 3 and one vertex has degree 1.

Q 13: Show that the number of vertices in a k-regular graph is even if k is odd.

Q 14: Let $v = [d_1, d_2, \ldots, d_n]$ and $w = [w_n, w_{n-1}, \ldots, w_2, w_1]$, where $w_i = n - 1 - d_i$. Show that v is graphic if and only if w is graphic.

Q 15: Show that there is no simple graph with six vertices of which the degrees of five vertices are 5, 5, 3, 2, and 1.

Q 16: Find k if $[8, k, 7, 6, 6, 5, 4, 3, 3, 1, 1, 1]$ is graphic.

Q 17: Show that an ordered sequence of nonincreasing numbers in which no two numbers are equal cannot be graphic.

Q 18: Show that in a simple graph, there are at least two vertices with equal degrees.

Q 19: Show that there exists a simple graph with 12 vertices and 28 edges such that the degree of each vertex is either 3 or 5. Draw this graph.

Q 20: Show that there exists a simple graph with seven vertices and 12 edges such that the degree of each vertex is 2 or 3 or 4.

Q 21: Prove that if u is a vertex of odd degree in connected graph G, then there exists a path from u to another vertex v of G where v also has odd degree.

Q 22: Let $d(u,v)$ denote the length of the shortest (u,v)-path in a connected graph G. Prove that d satisfies the triangle inequality: for any $u,v,w \in V(G) : d(u,v) + d(v,w) \geq d(u,w)$.

Q 23: Show that every simple graph with n vertices is isomorphic to a subgraph of the complete graph K_n.

Q 24: Prove that if two graphs G and G^* are isomorphic, then their respective ordered degree sequences should be the same.

Q 25: Show that if two graphs $G = (V,E)$ and $G^* = (V^*, E^*)$ are isomorphic, then $|V| = |V^*|$ and $|E| = |E^*|$.

Q 26: Show that two graphs G and $G*$ each having n vertices and m edges, need not be isomorphic.

Q 27: Show that two simple graphs are isomorphic if and only if their complements are isomorphic.

Q 28: Find a self-complementary graph G having four vertices.

Q 29: Find two self-complementary graphs having five vertices.

Q 30: Prove by induction that a complete graph with n vertices contains $n(n-1)/2$ edges.

Q 31: Compute the number of edges in K_n and in $K_{m,n}$.

Q 32: Use the fact that $\sum \delta(v) = 2|E|$ to find the size of K_n and $K_{m,n}$.

Q 33: Show that $(n-1) + (n-2) + (n-3) + \cdots + 1 + 0 = n(n-1)/2$

Q 34: Show that the number of vertices in a self-complementary graph is either $4k$ or $4k+1$, where k is a positive integer.

Q 35: Show that every graph has an even number of odd-degree vertices.

Q 36: Construct two nonisomorphic simple graphs with six vertices with degrees 1, 1, 2, 2, 3, and 3. What is the number of edges in each graph?

Q 37: Show that if G and G^* are isomorphic graphs, the degree of each vertex is preserved under the isomorphism.

Q 38: Show that it is not possible to have a group of seven people such that each person in the group knows exactly three other people in the group.

Q 39: Prove that in any group of six people, there will be either three people who know one another or three people who do not know one another.

Q 40: Show that if a bipartite graph $G = (\{V_1, V_2\}, E)$ is regular, then $|V_1| = |V_2|$.

Q 41: Construct two nonisomorphic cubic (i.e., 3-regular) graphs each with six vertices.

Q 42: Find the maximum number of edges in a bipartite graph.

Q 43: A k-cube is a simple connected graph with 2^k vertices. Each vertex is represented by a k-bit number. Let $d(u, v)$ be defined as the number of positions in which u and v have a different bit. Two vertices u and v are joined if and only if $d(u, v) = 1$. Show that a k-cube is a k-regular bipartite graph, and find the number of edges in a k-cube.

Q 44: Find the fewest vertices needed to construct a complete graph with at least 1000 edges.

Q 45: Test whether $[5, 4, 3, 3, 3, 3, 3, 2]$ is graphic. If it is graphic, draw a simple graph with this sequence as the degree sequence.

Q 46: Test whether $[6, 6, 5, 4, 3, 3, 1]$ is graphic.

Q 47: Find the complements of K_n and $K_{m,n}$.

Q 48: Show that if every edge in a graph joins an odd-degree vertex and an even-degree vertex, the graph is bipartite. Is the converse true?

Q 49: Show that every subgraph of a bipartite graph is also bipartite.

Q 50: Prove that for any graph G, $\kappa(G) \leq \lambda(G) \leq \min\{\delta(v)|v \in V(G)\}$

Q 51: Construct a graph for which $\kappa(G) < \lambda(G) < \min\{\delta(v)|v \in V(G)\}$ is strict.

Q 52: Provide an algorithm for checking whether an undirected graph G is connected.

Q 53: Prove that the Harary graph $H_{k,n}$ is k-connected.

Q 54: Prove that for a connected acyclic simple graph G with n vertices, $|E(G)| = n - 1$.

Q 55: Prove that for a plane graph G with n vertices, m edges, and r regions, we have that $n - m + r = 2$.

Q 56: Prove that for any connected simple planar graph G with $n \geq 3$ vertices and m edges, we have that $m \leq 3n - 6$

Q 57: Show that K_5 is nonplanar.

Q 58: Show that the complete bipartite graph $K_{3,3}$ is nonplanar.

Problems Chapter 3

Q 59: Show that for any simple undirected graph with m egdes there are 2^m possible orientations. What can we say about the number of orientations for nonsimple graphs?

Q 60: In Dijkstra's algorithm, we set $R_t(u) = S_t(u) \cup_{v \in S_t(u)} N(v)$, and later consider vertices from $R_t(u) \backslash S_t(u)$. Why can't we directly consider the set $\cup_{v \in S_t(u)} N(v)$?

Q 61: Apply Dijkstra's algorithm for vertex v_4 from Figure 3.4 and compute the weight of the resulting rooted tree $T(v_4)$. Find an alternative tree $T^*(v_4)$ that also gives shortest paths originating from v_4, but with a different weight, that is, $w(T(v_4)) \neq w(T^*(v_4))$.

Q 62: Change Dijkstra's algorithm so that it can be applied to weighted, strongly connected *directed* graphs.

Q 63: Let G be an undirected graph and $\mathcal{E} \stackrel{\text{def}}{=} \{E_1, \ldots, E_k\}$ a partitioning of its edge set. Let V_i be the collection of end points of edges from E_i. Prove that $\mathcal{E}$ is an edge coloring if and only if $|V_i| = 2 \cdot |E_i|$.

Q 64: A manufacturer of chemical goods is faced with the problem that certain goods cannot be stored at the same place due to the danger of unwanted reactions. What he seeks is a storage scheme such that goods that cannot be located at the same place are indeed separated. Provide a graph-theoretical model to solve this problem.

Q 65: Design a simple algorithm by which we can identify the components of a graph.

Q 66: Prove that there exists an orientation $D(G)$ for a connected undirected graph G that is strongly connected if and only if $\lambda(G) \geq 2$. In other words, G cannot be 1-edge-connected.

Q 67: Any orientation of the complete graph with vertex set $\{1, 2, \ldots, n\}$ is a tournament. A tournament is **transitive** if there is an arc from i to k whenever there is an arc from i to j and an arc from j to k for each i, j, and k. Construct both a transitive tournament with four vertices and one that is not transitive.

Q 68: Prove that in a digraph, the sum of the outdegrees of all the vertices is equal to the number of arcs, which is also equal to the sum of the indegrees of all the vertices.

Q 69: Prove that every walk in a graph between vertices v and w contains a path between v and w, and every directed walk from v to w in a digraph contains a directed path from v to w.

Q 70: Show that a graph G is bipartite if and only if $\chi(G) = 2$

Q 71: Construct the line graph of K_4.

Q 72: If v is the vertex in the line graph $L(G)$ that corresponds to the edge joining vertex x and vertex y in G, find the degree of v in $L(G)$.

Q 73: How many vertices does $L(K_n)$ have? And what about the number of edges?

Q 74: Let G be a simple graph with n vertices. Compute the number of edges in $L(G)$.

Q 75: Suppose G is a simple graph with five vertices with degrees 1, 2, 3, 3, and 3. Find the number of vertices and edges in $L(G)$.

Q 76: Show that there is no graph G such that $L(G) = K_{1,3}$.

Q 77: Construct an example to show that if $L(G)$ and $L(H)$ are isomorphic, it is not necessary that G and H are isomorphic.

Q 78: Show that

(a) a graph G is isomorphic to its line graph if and only if the degree of each vertex is 2

(b) the line graph of a connected graph G is (isomorphic to) K_n, if and only if G is (isomorphic to) $K_{1,n}$, when $n > 3$.

Q 79: Show that a digraph $D = (V, A)$ is strongly connected if and only if for every nonempty subset $X \subseteq V$ there exists an arc $\langle \overrightarrow{x, y} \rangle$ from a vertex $x \in X$ to a vertex $y \in V \backslash X$.

Q 80: Show that a vertex v of a connected graph is a cut vertex if and only if there exist two distinct vertices u and w such that every path between these two vertices passes through v.

Q 81: Show that any nontrivial graph has at least two vertices that are not cut vertices.

Q 82: Show that an edge of a connected graph is a cut edge if and only if there exist vertices v and w such that every path between these two vertices contains this edge.

Q 83: Show that an edge is a cut edge if and only if no cycle contains that edge.

Q 84: Show that in a graph with n vertices, the length of a path cannot exceed $(n - 1)$ and the length of a cycle cannot exceed n.

Q 85: Show that if a simple graph G with n vertices and m edges has k components, $m \leq \frac{1}{2}(n-k)(n-k+1)$.

Q 86: Find the minimum number of edges in a k-connected graph.

Q 87: Draw a k-connected graph with n vertices and m edges such that $2 \cdot m = n \cdot k$ when (a) $k = 1$ and (b) $k = 2$.

Q 88: Prove that a graph G is bipartite if and only if it contains no cycles of odd length.

Q 89: Show that the minimum number of time slots needed for the class-scheduling problem is the value of $\chi(G)$ of the associated graph G.

Q 90: Show that for any (simple, connected) graph G, $\chi(G) \leq \Delta(G) + 1$.

Q 91: Show that every planar graph G has a vertex v with $\delta(v) \leq 5$.

Q 92: Show that for any planar graph G, $\chi(G) \leq 5$.

Problems Chapter 4

Q 93: Consider a connected weighted graph G with two vertices of odd degree: u and v. Prove that by duplicating every edge on a minimum-weight (u, v)-path, we obtain a minimum-weight Eulerian graph.

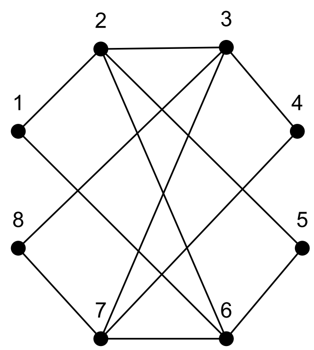
Q 94: A k-cube is a simple connected graph with 2^k vertices. Each vertex is represented by a k-bit number. Let $d(u, v)$ be defined as the number of positions in which u and v have a different bit. Two vertices u and v are joined if and only if $d(u, v) = 1$. Show that a k-cube is Hamiltonian.

Q 95: Show that a graph is Eulerian if and only if it is connected and if the set of its edges can be partitioned into a disjoint union of cycles.

Q 96: Prove that a connected graph G (with more than one vertex) has an Euler tour if and only if it has no vertices of odd degree.

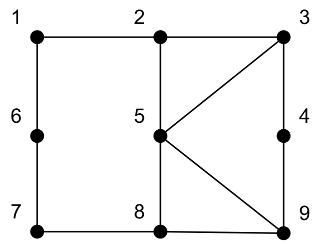
Q 97: Prove that a connected graph G (with more than one vertex) has an Euler trail if and only if it has exactly two vertices of odd degree. Moreover, the trail originates and ends in the vertices of odd degree.

Q 98: Using Fleury's algorithm, obtain an Euler tour for the following graph:



Q 99: If the number of odd-degree vertices in a connected graph $G = (V, E)$ is $2k$, show that the set E can be partitioned into k subsets such that the edges in each subset constitute a trail between two odd-degree vertices.

Q 100: Extend the following graph by adding a minimal number of edges such that the extended graph is simple and Eulerian.



Q 101: Prove that a weakly connected digraph is Eulerian if and only if the indegree of each vertex is equal to its outdegree.

Q 102: If every vertex in a graph G has even degree, no edge in that graph is a cut edge.

Q 103: Find all positive integers n such that K_n is Eulerian.

Q 104: Show that a digraph that has an Euler tour is a strongly connected digraph. Is the converse true?

Q 105: Consider a graph G with n vertices and m edges.

(a) Can G be Eulerian if n is even and m is odd?

(b) Can G be Eulerian if n is odd and m is even?

Q 106: If graph G is Hamiltonian, then for every proper nonempty subset $S \subset V(G)$, we have that $\omega(G - S) \leq |S|$.

Q 107: Prove that if G is a simple graph with $n = |V(G)|$ vertices, $n \geq 3$ and each vertex v has degree $\delta(v) \geq n/2$, then G is Hamiltonian.

Q 108: Let G be a non-Hamiltonian, connected graph. For every pair of nonadjacent vertices u and v, $\delta(u) + \delta(v) \geq k$, for some $k > 0$. Show that G contains a path of length k.

Q 109: If G is a connected graph with k odd-degree vertices, find the minimum number of trails in G such that every edge in the graph is an edge in exactly one of these trails.

Q 110: Suppose in a group of n people $(n > 3)$, any two of them together know all the other people in the group. Show that these n people can be seated around a circular table so that each person is seated between two acquaintances.

Q 111: Show that a directed graph D is Hamiltonian if and only if its transformed undirected version $\hat{D}$ is Hamiltonian.

Q 112: Show that a k-regular simple graph with $2k-1$ vertices is Hamiltonian.

Problems Chapter 5

Q 113: Prove that for any spanning tree T of a graph G and edge $e = \langle u, v \rangle \in E(G)$ that is not in T, $T + e$ contains a unique cycle.

Q 114: In Kruskal's algorithm, we select an edge $\hat{e}$ of the cycle C such that $\hat{e} \notin E(T_{opt})$, but $\hat{e} \in E(T)$. Prove that $\hat{e}$ indeed exists.

Q 115: Describe Dijkstra's algorithm for constructing a sink tree using pseudo-code, analogously to the description found in Chapter 3.

Q 116: Prove that for any connected graph G with n vertices and m edges, $n \leq m + 1$.

Q 117: Show by using a proof by induction that a tree with n vertices has exactly $n - 1$ edges.

Q 118: Prove that a connected graph G with n vertices and m edges for which $n = m + 1$, is a tree.

Q 119: Show that a graph G is a tree if and only if there exists exactly one path between every two vertices u and v.

Q 120: Prove that an edge e of a graph G is a cut edge if and only if e is not part of any cycle of G.

Q 121: Prove that a connected graph G is a tree if and only if every edge is a cut edge.

Q 122: Show that any tree with at least two vertices is bipartite.

Q 123: Show that a graph G is a tree if and only if it is acyclic and whenever any two vertices u and v in G are joined by an edge, the graph $G^* = G + \langle u, v \rangle$ has exactly one cycle.

Q 124: Prove that a graph is connected if and only if it has a spanning tree.

Q 125: Show that if a graph is disconnected, its complement is connected.

Q 126: Show that every tree of $n \geq 2$ vertices has at least two vertices having degree 1.

Q 127: Show that the sequence $\mathbf{d} = [d_1, d_2, \ldots, d_n]$ of positive integers, where $d_1 \leq d_2 \leq \ldots \leq d_n$ is the degree sequence of a tree with n vertices if and only if $\sum_{i=1}^{n} d_i = 2(n - 1)$.

Q 128: Show that the number of vertices in a binary tree is odd.

Q 129: Show that the number of terminal vertices in a binary tree with n vertices is $(n + 1)/2$.

Q 130: Let $\delta_{min}(G)$ denote the minimal vertex degree of graph G. Furthermore, let C_n denote the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{\langle v_1, v_2\rangle, \langle v_2, v_3\rangle, \ldots, \langle v_n, v_1\rangle\}$, i.e., a cycle of length n.

(a) Show that if T is a tree with n vertices and G is a graph with $\delta_{min}(G) \geq (n-1)$, T is isomorphic to a subgraph of G.

(b) Show that a tree with n vertices is isomorphic to a subgraph of the complement of C_{n+2}.

Q 131: Show that if $T_i = (V_i, E_i)$, where $i = 1, 2, ..., k$ are subtrees of $T = (V, E)$ such that every pair of subtrees have at least one vertex in common, the entire set of subtrees have a vertex in common.

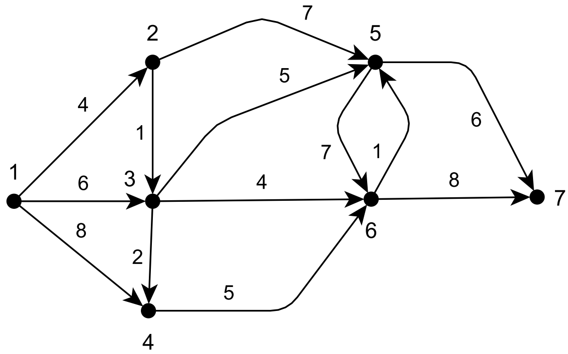
Q 132: If both G and its complement are trees, how many edges does G have?

Q 133: A forest is a graph consisting of k components, each component being a tree. How many edges does a forest of n vertices and k trees have?

Q 134: Show that if the degree of every non-leaf vertex in a tree is 3, the number of vertices in the tree is even.

Q 135: If the degree of each vertex in a graph is at least two, show that there is a cycle in the graph.

Q 136: Using Dijkstra's algorithm, find the sink tree rooted at vertex 7.

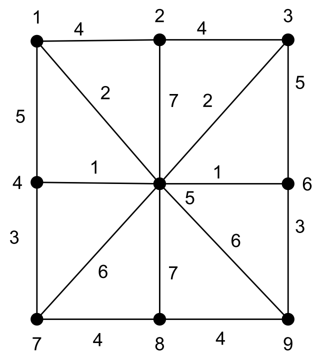


Q 137: List the edges of a sink tree rooted at vertex 1 of the network with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \langle 2, 5\rangle, \langle 3, 6\rangle, \langle 4, 5\rangle, \langle 5, 6\rangle\}$ with weights 4, 7, 3, 3, 2, 2, 3, and 2, respectively.

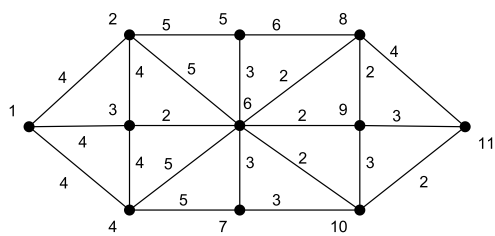
Q 138: If no two edge weights of a connected graph G are equal, show that G has a unique minimum spanning tree.

Q 139: Show that if a connected weighted graph G contains a unique edge e of minimum weight, e is an edge of every minimal spanning tree of G.

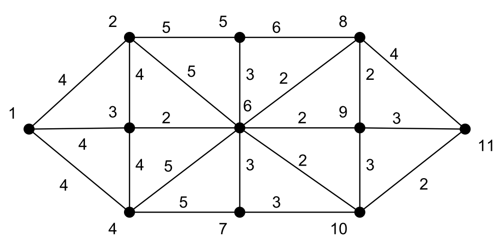
Q 140: Find the weight of a minimum spanning tree in the following graph, using Kruskal's algorithm.



Q 141: Obtain a minimum spanning tree for the following graph, using Kruskal's algorithm.



Q 142: Construct a *maximum* weight spanning tree for the following graph:



Problems Chapter 6

Q 143: Given a connected, simple undirected graph G with n vertices. Argue that a given vertex u can lie on at most $(n-1)(n-2)/2$ paths connecting other (distinct) vertices.

Q 144: Prove that the center of a tree is either a singleton set consisting of a unique vertex or a set consisting of two adjacent vertices.

Q 145: A path P between two distinct vertices in a connected graph G is a diametral path if there is no other path in G whose length is more than the length of P. Show that (a) every diametral path in a tree will pass through its central vertices, and (b) the center of a tree can be located once a diametral path in the tree is discerned.

Q 146: Given a connected, simple undirected graph G with n vertices. Argue that a given vertex u can lie on paths connecting exactly $(n-1)(n-2)/2$ different other (distinct) vertices.

Q 147: Show that the weighted clustering coefficient is identical to th clustering coefficient in an unweighted graph for the special case that all weights are equal to 1.

Q 148: Give an example of a simple, undirected graph G for which $CC(G) \neq \rho(G)$. Consider the case that all vertices of G have at least degree 2.

Q 149: Given an $ER(n,p)$ random graph. How many vertices can we expect to have vertex degree k?

Q 150: Prove that a triple is always connected.

Q 151: Explain why the giant cluster of a $ER(2000, 0.015)$ shrinks after removing more than 98% of the vertices.

Q 152: Show that $\sum_{i=1}^{m} i = \frac{1}{2}m(m+1)$, where we assume $m \geq 1$.

Q 153: Prove that $\sum_{k=m}^{n} \frac{1}{(k+1)(k+2)} = \frac{n-m+1}{(m+1)(2+n)}$.