Dept. Computer Science VU University Amsterdam	Graphs and Networks 01.03.2010

1 Show that every simple graph has two vertices of the same degree. *5pt*

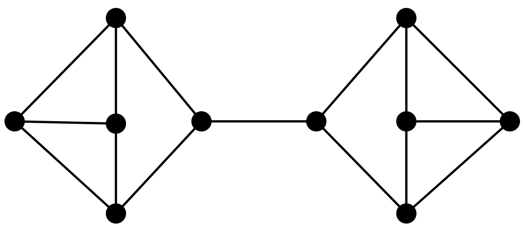
Answer: *Assume that the graph has n vertices. Each of those vertices is connected to either* $0,1,2,\ldots,n-1$ *other vertices. If any of the vertices is connected to* $n-1$ *vertices, then it is connected to all the others, so there cannot be a vertex connected to 0 others. Thus it is impossible to have a graph with n vertices where one is vertex has degree* 0 *and another has degree* $n-1$*. Thus the vertices can have at most* $n-1$ *different degrees, but since there are n vertices, at least two must have the same degree.*

2 The complement $\bar{G}$ of a graph G is the graph induced by the set of edges that are *not* in G. A self-complementary graph is one that is isomorphic with its complement. Find a self-complementary graph G having four vertices. *5pt*

Answer: *The number of edges in the complete graph with four vertices is 6. So if G is a self-complementary graph with four vertices, it should have three edges. Consider the following two versions of* K_4*, each with a thick-edged subgraph, and its thin-edged complement. Clearly, only the second one is self-complementary.*

3 Draw a 3-regular graph G for which $\lambda(G) = 1$, i.e., G has a cut edge. *5pt*

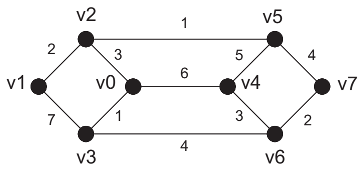
Answer:



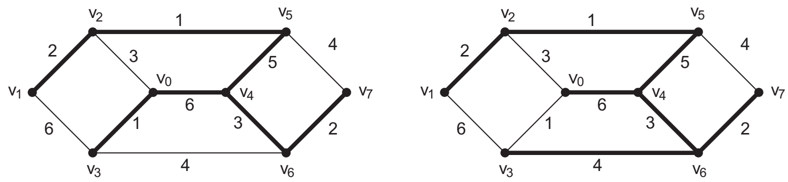
4 Show that every graph has an even number of odd-degree vertices. *4pt*

Answer: *Suppose the sum of the degrees of the odd vertices is k and the sum of the degrees of the even vertices is l. The number l is even, and the number* $k+l$*, being* $2 \cdot |E|$*, is also even. So k is necessarily even. If there are* $|V_{odd}|$ *odd-degree vertices, k is the sum of* $|V_{odd}|$ *odd numbers. So k is even.*

5 Apply Dijkstra's algorithm for vertex v_4 from the following figure and compute the weight of the resulting rooted tree $T(v_4)$. Find an alternative tree $T^*(v_4)$ that also gives shortest paths originating from v_4, but with a different weight, that is, $w(T(v_4)) \neq w(T^*(v_4))$. *8pt*



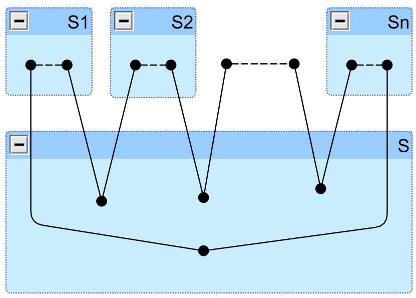
Answer: *The following figure shows two trees rooted at v_4, both containing shortest paths from v_4 to any other vertex. However, their weights are different, as can be readily observed.*



6 Let G be a Hamiltonian graph and $S \subset V(G)$, with $S \neq \emptyset$. Let C be a Hamilton cycle in G..

6a Explain why $\omega(C-S) \leq |S|$. *5pt*

Answer: *S divides G into n parts extra subgraphs $S_1,\ldots,S_n$ as shown in Figure 4.6. The smallest S does this as follows:*



In other words, $C-S$ may consist only of the components $S_1,S_2,\ldots,S_n$, which corresponds to $|S|$.

6b Explain why $\omega(G-S) \leq \omega(C-S)$. *3pt*

Answer: *It may very well be that there are still paths between subgraphs S_i and S_j, meaning that $\omega(G-S) \leq \omega(C-S)$.*

7 Show that in a graph with n vertices, the length of a path cannot exceed $(n-1)$ and the length of a cycle cannot exceed n. *5pt*

Answer: *If u and v are two vertices, the path between u and v can have at most $(n-2)$ distinct vertices. So the maximum length of the path is $(n-1)$. Likewise, the maximum length of a cycle is n.*

8 Prove that for any connected simple planar graph G with $n \geq 3$ vertices and m edges, we have that $m \leq 3n-6$ *10pt*

Answer: *Consider a region f in any plane graph of G. For any interior region, let $B(f)$ denote the number of edges by which f is enclosed, i.e., the length of its "border." Obviously, $B(f) \geq 3$ for any interior region. However, with $n \geq 3$ we also have that the exterior region is "bounded" by at least 3 edges. Therefore, if there are a total of r regions, then clearly $\sum B(f) \geq 3r$. On the other hand, it is not difficult to see that $\sum B(f)$ counts every edge in G once or twice, and hence $\sum B(f) \leq 2m$, so that we obtain $3r \leq \sum B(f) \leq 2m$, and thus $r \leq \frac{2}{3}m$. From Euler's formula we then derive that $m = n+r-2 \leq n+\frac{2}{3}m-2$, so that $m \leq 3n-6$.*

Grading: *The final grade is calculated by accumulating the scores per question (maximum: 50 points).*