Dept. Computer Science	**Graphs and Networks**
VU University Amsterdam	**26.03.2010**

BE SURE THAT YOUR HANDWRITING IS READABLE

Part I

1 Consider the following two graphs and determine for each whether it is (1) bipartite, (2) complete, (3) complete bipartite. *6pt*

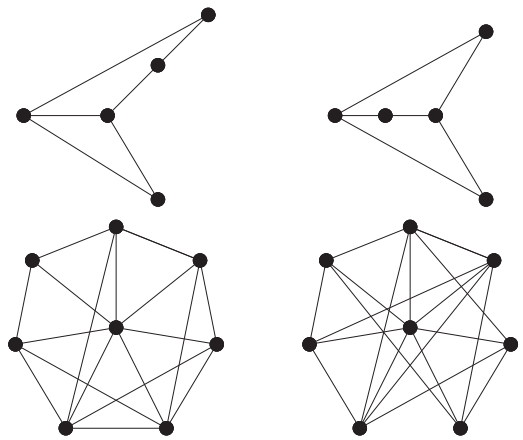
G1: $V = \{1,2,3,4,5,6\}$
$E = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,5\rangle, \langle 2,6\rangle, \langle 3,5\rangle, \langle 3,6\rangle, \langle 4,5\rangle, \langle 4,6\rangle\}$

G2: $V = \{1,2,3,4,5\}$
$E = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 2,5\rangle, \langle 3,4\rangle, \langle 3,5\rangle\}$

It is not hard to see that G_1 is isomorphic to $K_{3,3}$, with $V_1 = \{2,3,4\}$ and $V_2 = \{1,5,6\}$. In contrast, G_2 neither complete, nor bipartite, nor complete bipartite. To show that G_2 is not bipartite, we can, either exhaustively try all combinations of partitioning its vertex set, yet it is easier to simply identify an odd-length cycle (such as, for example, $[1,2,4,1]$.

2 Show that two graphs with the same degree sequence need not be isomorphic. *6pt*

Counter examples include the following:



3a Show that there is no simple graph with 12 vertices and 28 edges in which the degree of each vertex is either 3 or 4. *5pt*

3b Show that there is no simple graph with 12 vertices and 28 edges in which the degree of each vertex is either 3 or 6. *5pt*

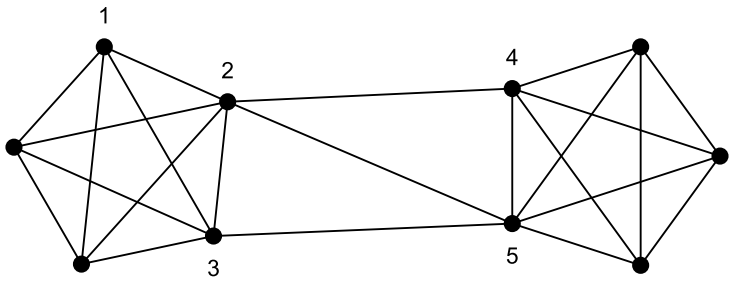
Suppose there is a graph with k vertices of degree 3 in the graph. For (a), if the remaining $(12-k)$ *vertices have all degree 4, the equation* $3k+4(12-k) = 56$ *gives* $k = -8$*, which is impossible. For (b), if the remaining* $(12-k)$ *vertices all have degree 6, the equation* $3k+6(12-k) = 56$ *gives* $k = 5\frac{1}{3}$*, which is also not possible.*

4 Let G be a simple, disconnected graph. Show that the complement $\bar{G}$ of G is connected. *8pt*

Consider two vertices u and v that belong to different components of G, and are thus adjacent in $\bar{G}$*. These two nodes cannot have a common neighbor in G, meaning that every vertex in* $\bar{G}$ *is adjacent to either u or v. This also means that every two vertices in* $\bar{G}$ *are connected through a path containing edge* $\langle u,v\rangle$.

5 Construct a graph for which $\kappa(G) < \lambda(G) < \min\{\delta(v) | v \in V(G)\}$. *8pt*

Consider the following graph. Clearly, $\delta(1) = 4$ *and is also the minimum vertex degree of G. Furthermore, the set* $\{\langle 2,4\rangle, \langle 2,5\rangle, \langle 3,5\rangle\}$ *forms a minimal edge cut of size 3, whereas the set of vertices* $\{2,5\}$ *forms a minimal vertex cut of size 2.*



6 If G is a bipartite graph with n vertices and m edges, prove that $m \leq (n^2)/4$. *6pt*

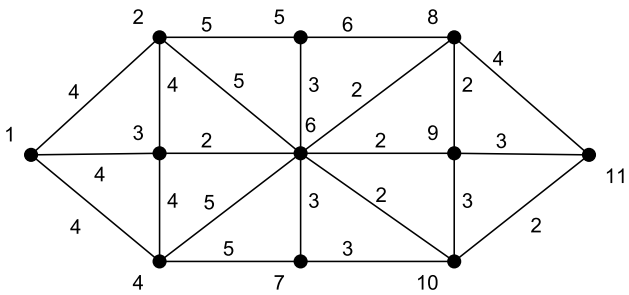
Say $G = (\{X,Y\},E)$ *with* $|X| = p$ *and* $|Y| = q$*. We know that* $p+q = n$*. Let* $p = v/2+d$ *and thus* $q = v/2-d$*. We also know that* $m \leq p \times q \leq (v/2)^2 - d^2 \leq (v/2)^2$*.*

7 Let $P = [u = v_1, v_2, \ldots, v_n = w]$ be a Hamilton (u,w)-path of a graph G, between two nonadajacent vertices u and w. Let T be the set $\{v_i | \exists \langle v_{i-1}, w\rangle \in E(G)\}$. How many elements does T have? Explain your answer. *6pt*

$|T| = \delta(w)$*, as all neighbors of w are traversed in P, and T contains exactly the successors of those neighbors.*

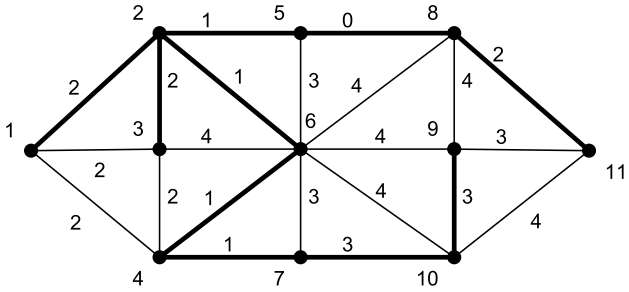
Part II

8 Using Kruskal's algorithm, construct a *maximum* weight spanning tree for the following graph. *6pt*



In this case, simply use Kruskal's algorithm on the same graph with each weight $w(e)$ *replaced by* $\max\{w(e) | e \in E(G)\} - w(e)$*. This gives a minimal spanning tree T which corresponds to a*

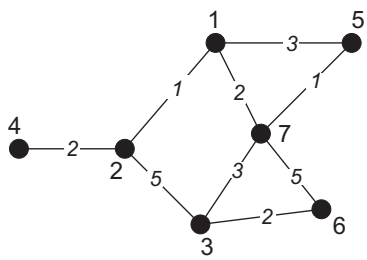
maximal-weight spanning tree with weight $\max\{w(e)|e \in E(G)\} - w(T)$.



9 Given an $ER(n,p)$ random graph. How many vertices can we expect to have vertex degree k? *8pt*

The probability that a vertex has degree k is equal to $p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$. *This means that we can expect a total of* $n \cdot p_k$ *vertices to have degree k.*

10 Compute for each vertex in the following graph G its eccentricity. Which vertices are in the center of G? Which vertex has maximal closeness? *10pt*



Vertex	*1*	*2*	*3*	*4*	*5*	*6*	*7*	$\varepsilon(u)$	$\bar{d}(u)$
1	*0*	*1*	*5*	*3*	*3*	*7*	*2*	*7*	*3.50*
2	*1*	*0*	*5*	*2*	*4*	*7*	*3*	*7*	*3.67*
3	*5*	*5*	*0*	*7*	*4*	*2*	*3*	*7*	*4.33*
4	*3*	*2*	*7*	*0*	*6*	*9*	*5*	*9*	*5.33*
5	*3*	*4*	*4*	*6*	*0*	*6*	*1*	*6*	*4.00*
6	*7*	*7*	*2*	*9*	*6*	*0*	*5*	*9*	*6.00*
7	*2*	*3*	*3*	*5*	*1*	*5*	*0*	*5*	*3.17*

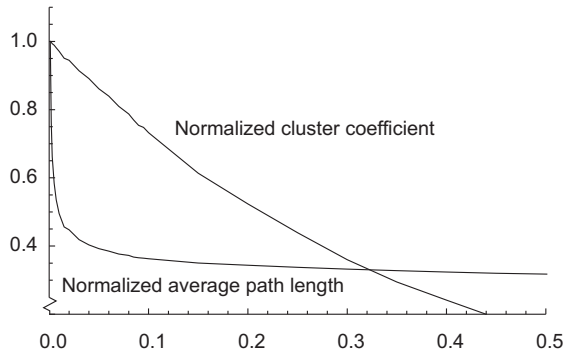
The center of G consists only of vertex 7: the one with the minimal eccentricity. The vertex with maximal closeness is also vertex 7.

11a Describe the construction of a Watts-Strogatz graph $WS(n,k,p)$. *6pt*

Consider a set of n vertices $\{v_1, v_2, \ldots, v_n\}$ *and an (even) number k.*

1. *Order the n vertices into a ring and connect each vertex to its first* $k/2$ *left-hand (clockwise) neighbors, and to its* $k/2$ *right-hand (counterclockwise) neighbors, leading to graph G.*
2. *With probability p, replace each edge* $\langle u,v \rangle$ *with an edge* $\langle u,w \rangle$ *where w is a randomly chosen vertex from* $V(G)$ *other than u, and such that* $\langle u,w \rangle$ *is not already contained in edge set of (the modified) G.*

11b For a $WS(n,k,p)$ graph, we know that the clustering coefficient and average path length evolve as a function of p as follows. What does this graph tell us? *6pt*



It tells us that a WS graph will continue to have a relatively high clustering coefficient while the average path length drops quickly for increasing values of p. In this sense, a WS graph exhibits what we would see in a small world.

12 The proximity prestige $p_{prox}(v)$ of a vertex is defined as follows, where $R^-(v)$ is the influence domain of vertex v. Explain in your own words the intuition behind this definition. *8pt*

$$p_{prox}(v) = \frac{|R^-(v)|/(n-1)}{\sum_{u \in R^-(v)} d(u,v)/|R^-(v)|}$$

In the definition, we consider each vertex that can possibly influence a vertex v, by considering all vertices that are connected to v by a directed path. Depending on how far a vertex is from v, its influence will change: the farther, the less influence. We therefore simply compute the average distance of all (potentially) influencing vertices to v, and subsequently take into account the fraction of all vertices that this set forms. As a consequence, the more and closer vertices are connected to v, the higher v's prestige will be.

13 What do we mean about a social network when we say that its associated graph is balanced? *6pt*

It means that the people who like each other have in common that they all dislike the same people. In this sense, the social network can be thought of as being stable.

Final grade: *(1) Add, per part, the total points. (2) Let T denote the total points for the midterm exam ($0 \leq T \leq 50$); $D1$ the total points for part I; $D2$ the total points for part II. The final number of points E is equal to* $\max\{T, D1\} + D2$.