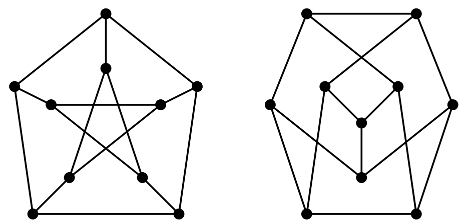
Dept. Computer Science | **Graphs and Networks**
VU University Amsterdam | **02.03.2009**

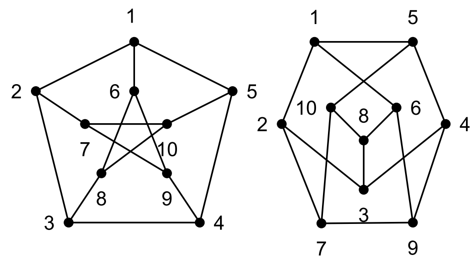
1 Show that the sequence $(7,6,5,4,3,3,2)$ is not graphic. Likewise, show that $(6,6,5,4,3,3,2)$ is not graphic. *4pt*

Answer: $(7,6,5,4,3,3,2)$ *can never be graphic because it is impossible to have a vertex with degree 7 in a simple graph on 7 vertices. If* $(6,6,5,4,3,3,2)$ *were graphic, then so would* $(5,4,3,2,2,1)$ *and* $(3,2,1,1,0)$ *and* $(1,0,0,0)$*, which is impossible. Also notice that* $\sum_v \delta(v)$ *for second sequence is odd, which is not possible for any sequence that would correspond to a graph.* ***Note:*** *For the Dutch version, the second sequence was* $(6,6,5,4,2,2,1)$.

2 Show that the following two graphs are isomorphic. *4pt*



Answer: *The following mapping between vertices does the job:*



Note: *Many students said the wto were isomorphic because they had the same degree sequence. As is also explicitly stated in the lecture notes, this is certainly not a sufficient condition for graph isomorphism.*

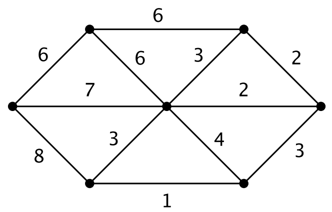
3 Show that in any group of two or more people, there are always two persons with exactly the same number of friends within that group. *Hint: for any graph, consider how many different vertex degrees are possible.* *6pt*

Answer: *Let G be a (simple) graph on n vertices and m edges, representing the group of people. Every vertex represents a person while an edge represents a friendship between two people. We need to show that for each vertex v in G, there exists another vertex with the same degree. The possible degrees for any vertex are* $0,1,2,\dots,n-1$. *Because* 0 *and* $n-1$ *cannot occur at the same time, we are faced with the situation that for n vertices there are at most* $n-1$ *possible vertex degrees. Hence, at least two vertices must have the same degree.*

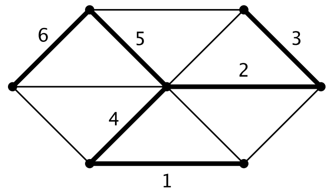
4 Show that for every undirected simple graph G, there exists an orientation D such that for each vertex $v \in V(D)$ we have $|\delta^+(v) - \delta^-(v)| \leq 1$. *8pt*

Answer: *Consider a longest trail P in G, from say vertex x to vertex y. Replace each edge* $\langle u,v\rangle \in E(P)$ *that is crossed when traversing P from x to y, with the arc* $\langle\overrightarrow{u,v}\rangle$. *Note that for every internal vertex in P, we will have as many incoming arcs as outgoing arcs, next to possibly a number of nondirected edges. If* $E(P) = E(G)$ *stop. Otherwise, consider the graph* $G' = G - P$ *and again consider a longest trail* P' *in* G'*, and direct the edges in the same way. Again, we will see that for any internal vertex of* P'*, the number of incoming arcs is the same as the number of outgoing arcs. In the end, we will see that for each vertex v in G,* $|\delta^+(v) - \delta^-(v)| \leq 1$.

5 Find a minimal spanning tree using Kruskal's algorithm for the following graph. Show the steps that you take. *5pt*



Answer: *One possibility is to add edges in the following order:*



6 Prove that for any tree with n vertices and m edges, $n = m + 1$. *6pt*

Answer: *The proof is by induction on the number of vertices. Clearly, when $n = 1$ there can be no edges and the statement is seen to hold. Then assume the statement holds for all trees with less than n vertices. Let H be a tree with $n \geq 2$ vertices, and edge $\langle u, v \rangle \in E(H)$. If we remove this edge, then the result will be two separate subgraphs G_1 and G_2, for otherwise $\langle u, v \rangle$ was part of a cycle. Both subgraphs are acyclic, each with less than n vertices, so that $|E(G_1)| = |V(G_1)| - 1$ and $|E(G_2)| = |V(G_2)| - 1$. Because we have not removed any vertices, we now have that*

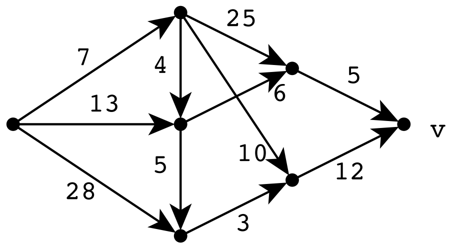
$$|E(H)| = |E(G_1)| + |E(G_2)| + 1 = |V(G_1)| - 1 + |V(G_2)| - 1 + 1 = n - 1$$

.

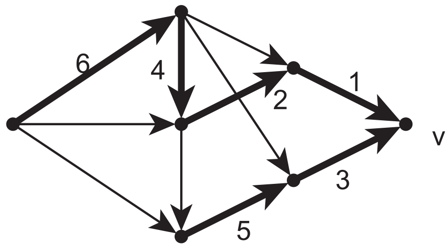
7 Prove that if a graph G is hamiltonian, then for every proper nonempty subset S of $V(G)$, the number of components in $G - S$ is less or equal than the number of elements in S. *5pt*

Answer: *Consider a Hamilton cycle C of G. If we consider any proper nonempty subset $S \subset V(G)$, then because every vertex is visited exactly once, the number of components in $C - S$ will be less or equal to $|S|$. Because C contains all vertices of G, we also have that $\omega(G - S) \leq \omega(C - S)$.*

8 Apply Dijkstra's algoritm to the following graph for constructing a sink tree rooted at v. Show the steps that you take. *7pt*



Answer: *The sink tree is obtained by adding vertices and edges in the following order:*



Grading: *The final grade is calculated by accumulating the scores per question (maximum: 45 points), and adding 5 bonus points. The maximum total is therefore 50 points.*