Tutorial Session
(week 14)

Rena Bakhshi

April 4-5, 2013
Goals:
- Revise parts of the main lecture
- Strengthen your graph theory skills (useful for the exam 😊)
- Have fun! 😊

Books:
- Maarten van Steen “Graph Theory And Complex Networks: An Introduction”
- Robin Wilson “Introduction to Graph Theory”

Communication by email:
- From your VU account (not ...@xs4all.nl!)
Homework Assignments

When?
- posted: Fridays (started from next week)
- deadline: Wednesdays at 11:59am (noon)

What?
- Single PDF and single NB (Mathematica) attachment
- Email to graphs.few@vu.nl
- Subject: <Vunet_ID>_<#>
  - E.g., for VUnet_ID abc123, and homework 1: subject = abc123_1
  - Wrong subject format: Notification email, no registration
  - Correct subject format: Confirmation email, registration
  - Wrong email: no mail response no feedback no grade!

**Adjacency Matrix**

**Definition**

The adjacency matrix of a *simple* graph on *n* vertices is an *n* × *n* matrix \( A = (a_{i,j}) \) in which the entry \( a_{i,j} \) is:

- \( =1 \), if there is an edge from vertex \( i \) to vertex \( j \)
- \( =0 \), if there is *no* edge from vertex \( i \) to vertex \( j \).

**Example**

![Diagram of a graph and its adjacency matrix]

**Observations**

- \( G \) is simple \( \iff \) \( A[i,j] \leq 1 \) and \( A[i,j] = 0 \).
**Adjacency Matrix**

**Exercise**

Give the adjacency matrix for each of the following graphs, and draw those graphs.

**G1:** \( V = \{1, 2, 3, 4, 5, 6\} \) and
\[
E = \{(1, 2), (1, 3), (1, 4), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6)\}
\]

**G2:** \( V = \{1, 2, 3, 4, 5\} \) and
\[
E = \{(1, 2), (1, 4), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5)\}
Adjacency Matrix
**Definition**

Let $G = (V, E)$ be a simple graph. The complement $\bar{G}$ of $G$ is a graph with the same vertex set $V$, but whose edge set consists of those edges not present in $G$: $\forall v \in V, \forall e : e \in E(\bar{G}) \text{ and } e \notin E$.

In other words, an edge $\langle v, u \rangle$ is in $\bar{G}$ if and only if it is not in $G$. 

**Complement**

**Definition**

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![Graph $G$ and its complement $\bar{G}$]
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$G$ $\bar{G}$
**Complement**

**Definition**

Let $G = (V, E)$ be a simple graph. The complement $\tilde{G}$ of $G$ is a graph with the same vertex set $V$, but whose edge set consists of those edges not present in $G$: $v \in V, \forall e : e \in E(\tilde{G})$ and $e \notin E$.

In other words, an edge $\langle v, u \rangle$ is in $\tilde{G}$ if and only if it is not in $G$.

Superimposing any graph of $n$ nodes with its complement, gives the complete graph $K_n$. 
**Complement**

**Definition**

Let $G = (V, E)$ be a simple graph. The complement $\bar{G}$ of $G$ is a graph with the same vertex set $V$, but whose edge set consists of those edges not present in $G$: $\forall v \in V, \forall e : e \in E(\bar{G})$ and $e \notin E$.

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Superimposing any graph of $n$ nodes with its complement, gives the complete graph $K_n$. 
Complement

Find complements of these two graphs:
COMPLEMENT

Find complements of these two graphs:

SOLUTION
Degree sequence

Theorem (Havel-Hakimi)

An ordered degree sequence $s$ is graphic, if and only if $s^*$ is also graphic.

$$s = [k, d_1, d_2, \ldots, d_{n-1}]$$

$$s^* = [d_1 - 1, d_2 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_{n-1}]$$

and $k \geq d_i \geq d_{i+1}$.

Example

$[3, 2, 2, 1]$
Degree sequence

Theorem (Havel-Hakimi)

An ordered degree sequence $s$ is graphic, if and only if $s^*$ is also graphic.

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and $k \geq d_i \geq d_{i+1}$.

Example

\[ [3,2,2,1] \rightarrow [1,1,0] \]
Degree sequence

Theorem (Havel-Hakimi)

An ordered degree sequence \( s \) is graphic, if and only if \( s^* \) is also graphic.

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s = [k, d_1, d_2, \ldots, d_{n-1}]
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\[
s^* = [d_1 - 1, d_2 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_{n-1}]
\]

and \( k \geq d_i \geq d_{i+1} \).

Example

\[ [3,2,2,1] \rightarrow [1,1,0] \rightarrow [0,0] \]
DEGREE SEQUENCE

THEOREM (HAVEL-HAKIMI)

An ordered degree sequence $s$ is graphic, if and only if $s^*$ is also graphic.

$$s = [k, d_1, d_2, \ldots, d_{n-1}]$$
$$s^* = [d_1 - 1, d_2 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_{n-1}]$$

and $k \geq d_i \geq d_{i+1}$.

EXAMPLE

$$[3, 2, 2, 1] \rightarrow [1, 1, 0] \rightarrow [0, 0]$$

Draw a graph with this sequence.
**Degree sequence**

**Theorem (Havel-Hakimi)**

An ordered degree sequence $s$ is graphic, if and only if $s^*$ is also graphic.

$$s = [k, d_1, d_2, \ldots, d_{n-1}]$$

$$s^* = [d_1 - 1, d_2 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_{n-1}]$$

and $k \geq d_i \geq d_{i+1}$.

**Exercises**

Are these sequences graphic:

- $[3,2,1,1]$;  
- $[2,2,2,2]$;  
- $[5,4,4,3,1,1]$;  
- $[3,3,2,2,1,1]$;  
- $[4,4,1,1,1,1,1,1]$  

Apply Havel-Hakimi theorem.
**Degree sequence**

**Solutions**

- [2, 2, 2, 2, 2]
- [4, 4, 1, 1, 1, 1, 1]
- [3, 2, 2, 1]
- [3, 3, 2, 2, 1, 1]
Isomorphism

Definition

$G_1$ and $G_2$ are isomorphic if there is a one-to-one mapping $\phi : V_1 \rightarrow V_2$ such that for each edge $e_1 \in E_1$ with $e_1 = \langle v, u \rangle$ there is a unique edge $e_2 \in E_2$ with $e_2 = \langle \phi(v), \phi(u) \rangle$. 

Example

$G_1$ and $G_2$ Solution

$\phi = \{ 1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 4, 4 \rightarrow 3 \}$.
**Isomorphism**

**Definition**

$G_1$ and $G_2$ are **isomorphic** if there is a one-to-one mapping $\phi : V_1 \rightarrow V_2$ such that for each edge $e_1 \in E_1$ with $e_1 = \langle v, u \rangle$ there is a unique edge $e_2 \in E_2$ with $e_2 = \langle \phi(v), \phi(u) \rangle$.

Informally, two graphs are isomorphic if they contain the same number of vertices, connected in the same way.
**Isomorphism**

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Iso-Cube Demo!
**Isomorphism**

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**Example**

![Diagram](image-url)
**Isomorphism**

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**Example**

![Graph $G_1$ and $G_2$](image)

**Solution**

$$\phi = \{1 \rightarrow 1, \ 2 \rightarrow 2, \ 3 \rightarrow 4, \ 4 \rightarrow 3\}.$$
**Isomorphism**

**Definition**

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Example

$G_1$ $G_2$

Solution

$\phi = \{1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 2, 4 \rightarrow 5, 5 \rightarrow 3\}$. 


Isomorphism

Definition

$G_1$ and $G_2$ are isomorphic if there is a one-to-one mapping $\phi : V_1 \rightarrow V_2$ such that for each edge $e_1 \in E_1$ with $e_1 = \langle v, u \rangle$ there is a unique edge $e_2 \in E_2$ with $e_2 = \langle \phi(v), \phi(u) \rangle$.

Example

$G_1$ and $G_2$ are isomorphic with $\phi = \{1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 2, 4 \rightarrow 5, 5 \rightarrow 3\}$.
**ISOMORPHISM**

**Definition**

$G_1$ and $G_2$ are isomorphic if there is a one-to-one mapping $\phi : V_1 \rightarrow V_2$ such that for each edge $e_1 \in E_1$ with $e_1 = \langle v, u \rangle$ there is a unique edge $e_2 \in E_2$ with $e_2 = \langle \phi(v), \phi(u) \rangle$.

**Example**

![Diagram showing $G_1$ and $G_2$]

**Solution**

$$\phi = \{1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 2, 4 \rightarrow 5, 5 \rightarrow 3\}.$$
**Isomorphism**

**Definition**

$G_1$ and $G_2$ are **isomorphic** if there is a one-to-one mapping $\phi : V_1 \rightarrow V_2$ such that for each edge $e_1 \in E_1$ with $e_1 = \langle v, u \rangle$ there is a unique edge $e_2 \in E_2$ with $e_2 = \langle \phi(v), \phi(u) \rangle$
**Isomorphism**

**Exercise**

Which of the pairs of graphs are isomorphic? (If yes, find \( \phi \); if not, why).

**Solutions**

(a) Yes: \( 1 \rightarrow 4, \ 2 \rightarrow 2, \ 3 \rightarrow 3, \ 4 \rightarrow 1, \ 5 \rightarrow 5 \).
**ISOMORPHISM**

**EXERCISE**

Which of the pairs of graphs are isomorphic? (If yes, find \( \phi \); if not, why).

**SOLUTIONS**

(a) Yes: \( 1 \rightarrow 4, 2 \rightarrow 2, 3 \rightarrow 3, 4 \rightarrow 1, 5 \rightarrow 5 \).
(b) Yes: \( 1 \rightarrow 4, 2 \rightarrow 5, 3 \rightarrow 1, 4 \rightarrow 2, 3 \rightarrow 5 \).
**Isomorphism**

**Exercise**

Which of the pairs of graphs are isomorphic? (If yes, find $\phi$; if not, why).

![Graphs](image)

(c) 

(d)

**Solutions**
**Isomorphism**

**Exercise**

Which of the pairs of graphs are isomorphic? (If yes, find $\phi$; if not, why).

(c) (d)

**Solutions**

(c) No: because node 2 has degree 4 in the left graph.

(d) No: because node 5 has degree 5 in the left graph.
**Self-complementary Graphs**

**Definition**
A graph is called self-complementary if it is isomorphic to its complement.

**Exercise**
Find all self-complementary graphs with 4 nodes.
**Self-complementary Graphs**

**Definition**
A graph is called **self-complementary** if it is **isomorphic to its complement**.

**Exercise**
Find all self-complementary graphs with 4 nodes.

**Answer**
- Let us first think how many edges a self-complementary graph $G$ of 4 nodes should have.
SELF-COMPLEMENTARY GRAPHS

Definition
A graph is called self-complementary if it is isomorphic to its complement.

Exercise
Find all self-complementary graphs with 4 nodes.

Answer
- Let us first think how many edges a self-complementary graph $G$ of 4 nodes should have.
- If $G$ has $k$ edges, $\bar{G}$ also has $k$ (because they are isomorphic).
SELF-COMPLEMENTARY GRAPHS

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EXERCISE

Find all self-complementary graphs with 4 nodes.

ANSWER

- Let us first think how many edges a self-complementary graph $G$ of 4 nodes should have.
- If $G$ has $k$ edges, $\bar{G}$ also has $k$ (because they are isomorphic).
- But the edges of $G$ plus the edges of $\bar{G}$ should be equal to the number of edges in the complete graph $K_4$. 


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- But the edges of $G$ plus the edges of $\bar{G}$ should be equal to the number of edges in the complete graph $K_4$.
- So: $|E(G)| + |E(\bar{G})| = |E(K_4)|$
**Self-complementary Graphs**

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- But the edges of $G$ plus the edges of $\bar{G}$ should be equal to the number of edges in the complete graph $K_4$.
- So: $|E(G)| + |E(\bar{G})| = |E(K_4)|$
- $\Rightarrow k + k = 6$
- $\Rightarrow k = 3$
Self-complementary Graphs

**Definition**
A graph is called *self-complementary* if it is *isomorphic to its complement*.

**Exercise**
Find all self-complementary graphs with 4 nodes.

**Answer**
- Here we list all graphs with 4 nodes and 3 edges (we do not put the isomorphic ones):

  ![Graphs](image-url)
**Self-complementary Graphs**

**Definition**

A graph is called self-complementary if it is isomorphic to its complement.

**Exercise**

Find all self-complementary graphs with 4 nodes.

**Answer**

- Here we list all graphs with 4 nodes and 3 edges (we do not put the isomorphic ones):

  ![Graphs](image)

- Clearly, only the left one is self-complementary.
**Proof**

**Tips**

- What you are going to prove.
- What method of proof you will to use.
- Make sure each statement follows logically from the previous.
- Be very careful in how you phrase each sentence.
- Define your notations and follow them.

**Methods**

- Direct proof;
- By construction;
- By contradiction;
- By induction.
Direct Proof

A task of the form:

“For some statements $p$ and $q$, $\forall x \in X$ if $p(x)$ is true then $q(x)$ is true”

Assume $p$, and prove that $q$ holds for all values of $x$ using known axioms, facts and theorems.

Example

Show that the degree sequence \[2, 2, 2, 2, 2\] is graphical.

Proof.

Using Havel-Hakimi theorem, \[2, 2, 2, 2, 2\] → \[2, 2, 1, 1\] → \[1, 0\] → \[0, 0\].

The degree sequence \[0, 0\] is graphical. Hence, the original sequence \[2, 2, 2, 2, 2\] is graphical too.
Direct Proof

A task of the form:

“For some statements $p$ and $q$, $\forall x \in X$ if $p(x)$ is true then $q(x)$ is true”

Assume $p$, and prove that $q$ holds for all values of $x$ using known axioms, facts and theorems.

Example

Show that the degree sequence $[2, 2, 2, 2, 2]$ is graphical.
**Direct Proof**

A task of the form:

“For some statements \( p \) and \( q \), \( \forall x \in X \) if \( p(x) \) is true then \( q(x) \) is true”

Assume \( p \), and prove that \( q \) holds for all values of \( x \) using known axioms, facts and theorems.

**Example**

Show that the degree sequence \([2, 2, 2, 2, 2]\) is graphical.

**Proof.**

Using Havel-Hakimi theorem,

\[
[2, 2, 2, 2, 2] \rightarrow [2, 2, 1, 1] \rightarrow [1, 1, 0] \rightarrow [0, 0].
\]

The degree sequence \([0, 0]\) is graphical. Hence, the original sequence \([2, 2, 2, 2, 2]\) is graphical too.
**By construction**

**Example**

A task of the form:

\[
\text{There is/exists an } x \text{ such that } p(x) \text{ true.}
\]

**Example**

Show that the degree sequence \([2, 2, 2, 2, 2]\) is graphical.
**By construction**

**Example**

A task of the form:

*There is/exists an* \( x \) *such that* \( p(x) \) *true.*

**Example**

Show that the degree sequence \([2, 2, 2, 2, 2]\) is graphical.

**Counter-example**

Usually, of the form:

*For all* \( x \in X \), \( p(x) \) *is true.*

To disprove it, one example of \( x \in X \) (called counter-example) is sufficient.

**Example**

For any graph, there is one vertex/node in a graph that has no edges.
**Isomorphism**

**Exercise**
Show that two graphs of the same number of vertices and the same degrees on corresponding vertices are not necessarily isomorphic.
**Isomorphism**

**Exercise**
Show that two graphs of the *same number of vertices* and the *same degrees* on corresponding vertices are *not necessarily isomorphic*.

**Answer**
We show this by *construction*:
**Isomorphism**

**Exercise**
Show that two graphs of the same number of vertices and the same degrees on corresponding vertices are not necessarily isomorphic.

**Answer**
We show this by construction:
Proof Methods

By contradiction
An indirect form of proof: proving $p$ by assuming $p$ is false.

Example
The degree sequence $[3, 2, 1, 1]$ is not graphical.
Proof Methods

By contradiction
An indirect form of proof: proving $p$ by assuming $p$ is false.

Example
The degree sequence $[3, 2, 1, 1]$ is not graphical.

Proof.
We prove it by contradiction.
**Proof Methods**

**By contradiction**
An indirect form of proof: proving $p$ by assuming $p$ is false.

**Example**
The degree sequence $[3, 2, 1, 1]$ is *not* graphical.

**Proof.**
We prove it by *contradiction*.
**Assumption:** Suppose the degree sequence is graphical.
Proof Methods

By contradiction
An indirect form of proof: proving \( p \) by assuming \( p \) is false.

Example
The degree sequence \([3, 2, 1, 1]\) is not graphical.

Proof.
We prove it by contradiction.
Assumption: Suppose the degree sequence is graphical.

- Then by applying Havel-Hakimi theorem we should reach eventually sequence which is also graphical.
Proof Methods

By contradiction
An indirect form of proof: proving $p$ by assuming $p$ is false.

Example
The degree sequence $[3, 2, 1, 1]$ is not graphical.

Proof.
We prove it by contradiction.
Assumption: Suppose the degree sequence is graphical.
- Then by applying Havel-Hakimi theorem we should reach eventually sequence which is also graphical.
- This means, $[3, 2, 1, 1] \rightarrow [1, 0, 0] \rightarrow [-1, 0]$. 
**Proof Methods**

**By contradiction**
An indirect form of proof: proving $p$ by assuming $p$ is false.

**Example**
The degree sequence $[3, 2, 1, 1]$ is *not* graphical.

**Proof.**
We prove it by *contradiction*.

**Assumption:** Suppose the degree sequence is graphical.

- Then by applying Havel-Hakimi theorem we should reach eventually sequence which is also graphical.
- This means, $[3, 2, 1, 1] \rightarrow [1, 0, 0] \rightarrow [-1, 0]$.
- But the degree cannot be negative, which implies the original assumption was *false*. 
Proof Methods

Exercise

Show that there is no simple graph with
- 12 vertices and
- 28 edges

in which the degree of each vertex is either 3 or 4.
Proof Methods

Exercise
Show that there is no simple graph with

- 12 vertices and
- 28 edges

in which the degree of each vertex is either 3 or 4.

Proof.
We prove it by contradiction.
**Exercise**

Show that there is no simple graph with

- 12 vertices and
- 28 edges

in which the **degree of each vertex is either 3 or 4**.

**Proof.**

We prove it by **contradiction**.

**Assumption:** Suppose that such a graph **exists**, and it has $k$ vertices of degree 3.
**Proof Methods**

**Exercise**

Show that there is **no** simple graph with

- **12 vertices** and
- **28 edges**

in which the **degree of each vertex is either 3 or 4**.

**Proof.**

We prove it by **contradiction**.

**Assumption:** Suppose that such a graph **exists**, and it has \( k \) vertices of degree 3.

- The remaining \((12 - k)\) vertices have all degree 4.
Intro

Proof Methods

Exercise

Show that there is no simple graph with

- 12 vertices and
- 28 edges

in which the degree of each vertex is either 3 or 4.

Proof.

We prove it by contradiction.

Assumption: Suppose that such a graph exists, and it has $k$ vertices of degree 3.

- The remaining $(12 - k)$ vertices have all degree 4.
- We know that, the sum of the degrees is $\sum_{v \in V} \delta(v) = 2|E(G)|$.
- Then, $3 \cdot k + 4 \cdot (12 - k) = 28 \cdot 2 = 56$
**Proof Methods**

**Exercise**
Show that there is no simple graph with
- **12 vertices** and
- **28 edges**
in which the degree of each vertex is either 3 or 4.

**Proof.**
We prove it by **contradiction**.
**ASSUMPTION:** Suppose that such a graph **exists**, and it has **$k$ vertices** of degree 3.
- The remaining $(12 - k)$ vertices have all degree 4.
- We know that, the sum of the degrees is $\sum_{v \in V} \delta(v) = 2|E(G)|$.
- Then, $3 \cdot k + 4 \cdot (12 - k) = 28 \cdot 2 = 56$
- Solving this, gives $k = -8$. Impossible, which implies the original assumption was **false**.
Proof Methods

Powerful method of proof based on Induction Principle. Used in many areas of mathematics and (theoretical) computer science.

The Induction Principle

Let \( p(n) \) be a statement that involves \( n \in \mathbb{N} \) (i.e., \( n = 1, 2, \ldots \)). Then \( p(n) \) is true for all \( n \) if:

- \( p(1) \) is true, and
- \( p(n) \) implies \( p(n + 1) \) is true for all natural numbers \( n \).
Intro

Proof Methods

By (Mathematical) Induction

A task of the form:

For all $n \geq a$, $p(n)$.

Proof by induction consists of:

- **Base case**: assuming $n = a$, show that $p(a)$ holds.
- **Inductive Hypothesis**: Assume $p(n)$ holds.
- **Inductive step**: If $p(n)$ holds, show $p(n + 1)$ holds.
Example

Show that for $n \in \mathbb{N}$ with $n \geq 1$ we have:

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$
Example

Show that for $n \in \mathbb{N}$ with $n \geq 1$ we have:

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

Proof.

Basic case We test for $n = 1$. Indeed, we have $2^1 = 2^2 - 2$. 
Proof Methods

Example
Show that for \( n \in \mathbb{N} \) with \( n \geq 1 \) we have:

\[
2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2
\]

Proof.
Basic case We test for \( n = 1 \). Indeed, we have \( 2^1 = 2^2 - 2 \).
Hypothesis We assume that the formula holds for \( n = k \):
\[
\sum_{i=1}^{k} 2^i = 2^{k+1} - 2.
\]
Introduction

Proof Methods

Example

Show that for $n \in \mathbb{N}$ with $n \geq 1$ we have:

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

Proof.

Basic case
We test for $n = 1$. Indeed, we have $2^1 = 2^2 - 2$.

Hypothesis
We assume that the formula holds for $n = k$:

$$\sum_{i=1}^{k} 2^i = 2^{k+1} - 2.$$ 

Induction step
We prove that for $n = k + 1$, $\sum_{i=1}^{k+1} 2^i = 2^{k+2} - 2$. 


**Proof Methods**

**Example**
Show that for \( n \in \mathbb{N} \) with \( n \geq 1 \) we have:

\[
2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2
\]

**Proof.**

**Basic case** We test for \( n = 1 \). Indeed, we have \( 2^1 = 2^2 - 2 \).

**Hypothesis** We assume that the formula holds for \( n = k \):

\[
\sum_{i=1}^{k} 2^i = 2^{k+1} - 2.
\]

**Induction step** We prove that for \( n = k + 1 \), \( \sum_{i=1}^{k+1} 2^i = 2^{k+2} - 2 \).

- This can be proved as follows:
Proof Methods

Example
Show that for $n \in \mathbb{N}$ with $n \geq 1$ we have:

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

Proof.

Basic case We test for $n = 1$. Indeed, we have $2^1 = 2^2 - 2$.

Hypothesis We assume that the formula holds for $n = k$:

$$\sum_{i=1}^{k} 2^i = 2^{k+1} - 2.$$ 

Induction step We prove that for $n = k + 1$, $\sum_{i=1}^{k+1} 2^i = 2^{k+2} - 2$.

This can be proved as follows:

$$\sum_{i=1}^{k+1} 2^i = \sum_{i=1}^{k} 2^i + 2^{k+1} \quad \text{assumption}$$

$$= (2^{k+1} - 2) + 2^{k+1} = 2^{k+2} - 2.$$
**Exercise**

Use **induction** again to show that for $n \in \mathbb{N}$ with $n \geq 1$, the number $7^n - 1$ is divisible by 6.
**Proof Methods**

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Use **induction** again to show that for $n \in \mathbb{N}$ with $n \geq 1$, the number $7^n - 1$ is divisible by 6.

**Proof.**

**Basic case** We test for $n = 1$: $7^1 - 1 = 6$, which is clearly divisible by 6.
Proof Methods

Exercise

Use induction again to show that for $n \in \mathbb{N}$ with $n \geq 1$, the number $7^n - 1$ is divisible by 6.

Proof.

Basic case We test for $n = 1$: $7^1 - 1 = 6$, which is clearly divisible by 6.

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**Proof Methods**

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- Indeed, $7^{k+1} - 1 = 7^{k+1} - 7 + 6 = 7 \cdot 7^k - 7 \cdot 1 + 6 = 7^{k+1} - 7 \cdot 1 + 6 = 7 \cdot (7^k - 1) + 6$
**Proof Methods**

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- Indeed, $7^{k+1} - 1 = 7^{k+1} - 7 + 6 = 7 \cdot 7^k - 7 \cdot 1 + 6$
  
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- $(7^k - 1)$ is divisible by 6, according to hypothesis.
**Exercise**

Use **induction** again to show that for $n \in \mathbb{N}$ with $n \geq 1$, the number $7^n - 1$ is divisible by 6.

**Proof.**

**Basic case** We test for $n = 1$: $7^1 - 1 = 6$, which is clearly divisible by 6.

**Hypothesis** We assume that the property holds for $n = k$, that is, $7^k - 1$ is divisible by 6.

**Induction step** We prove that it also holds for $n = k + 1$, that is, $7^{k+1} - 1$ is also divisible by 6.

- Indeed, $7^{k+1} - 1 = 7^{k+1} - 7 + 6 = 7 \cdot 7^k - 7 \cdot 1 + 6$
  - $= 7^{k+1} - 7 \cdot 1 + 6 = 7 \star (7^k - 1) + 6$
  - $(7^k - 1)$ is divisible by 6, according to hypothesis.
  - Hence, $7 \star (7^k - 1) + 6$ clearly divisible by 6, being the sum of two entities that are both divisible by 6.
See you next week!